Pinning transitions in *d*-dimensional Ising ferromagnets

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The pinning of a domain wall by an external potential is studied in the solid-on-solid limit of the d-dimensional inhomogeneous Ising ferromagnet. Analyzing a field theoretic generalization of this model, we show that, if the pinning potential is applied near the edge of the system, a localization-delocalization transition occurs for all $\frac{5}{3} < d \leq 3$. Results for singular behavior of the surface tension, correlation length, and interface moments are derived.

An interesting new type of domain-wall pinning transition in inhomogeneous two-dimensional Ising ferromagnets has recently been reported by Abraham.[1] In his model it is energetically favorable for the domain wall to pass through a row of weakened bonds located near one edge of the system. At sufficiently low temperatures the interface is found to be "pinned" to this row of defect bonds. The magnetization profile has a zero at a finite distance from the free surface and its width is finite. However, at a temperature T_p less than the two-dimensional Ising critical temperature he found a sharp delocalization transition. The zero intercept of the magnetization profile was found to diverge as $(T_p - T)^{-1}$ as T_p is approached from below. Above T_p the interface is no longer localized and simple random-walk arguments for the interface profile were found to apply. Associated with this transition he found a jump discontinuity in the second derivative of the interface tension (domain-wall specific heat).

Subsequently, several authors have studied the pinning of one-dimensional interfaces using a variety of solid-on-solid (SOS) models.[2–7] The canonical form of the configuration energy in these models is

$$\beta E\{f\} = \beta \sum_i |f_{i+1} - f_i| + \sum_i \beta V(f_i) \quad , \tag{1}$$

where $f_i \geq 0$ denotes the perpendicular distance of the interface from point i on the edge of the system, $\beta^{-1} = k_B T$, and $V(f) < 0$ is a short-range pinning potential localized near $f = 0$. The partition function is obtained by summing $e^{-\beta E\{f\}}$ over configurations $\{f\}$. It has been found that it is not necessary to restrict the height variables f_i to discrete values when evaluating the partition function; in fact, pinning transitions belonging to the same universality class have been obtained when the f_i vary continuously in the interval $0 \leq f_i < \infty$.[2,3,7] A typical pinning potential $\beta V\{f\}$ is shown by the broken curve in Fig. 1.

In this Communication we present results for interface pinning transitions induced by short-range pinning potentials in inhomogeneous Ising ferromagnets in arbitrary dimension. The model we consider is a field theoretic generalization of the SOS model described above. The Lagrangian (5) for this model proves to be remarkably simple to handle and is fully renormalized by normal ordering. This allows us to derive essentially exact results for the system's critical behavior. As far as we know, ours are the first quantitative results concerning the pinning transition of a D-dimensional interface embedded in a $d(=D+1)$-dimensional inhomogeneous Ising ferromagnet.[8]

Our starting point is the D-dimensional continuum generalization of (1):

$$\beta H = \int d^D x \left[\tfrac{1}{2}(\nabla f)^2 + \beta V(f/\sqrt{\beta})\right] \quad , \tag{2}$$

where a factor $\sqrt{\beta}$ has been absorbed in f. The partition function for this model is

$$Z = \int_0^\infty d\{f\} e^{-\beta H} \quad , \tag{3}$$

where the functional integration is restricted to

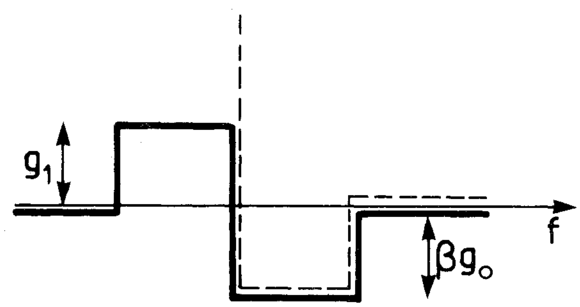

FIG. 1. Pinning potentials for various solid-on-solid models. Dashed curve: potential $\beta V(f)$ of Eq. (1). Full curve: potential $\beta V(f) + U(f)$ of Eq. (4).

$\{f(x)\} \geq 0$. In other words, at $f=0$ there is an infinite repulsive (temperature-independent) barrier which restricts the height variables to positive values.

Explicit calculations for $D=1$ show, however, that the character of the pinning transition is not changed if this infinite repulsive barrier is replaced by a finite one.[9] The restriction $f(x) \geq 0$ in (3) may therefore be dropped if, instead of (2), we consider the Lagrangian

$$\mathcal{L} = \beta H + U = \int d^D x \left[\tfrac{1}{2}(\nabla f)^2 + \beta V(f/\sqrt{\beta}) + U(f/\sqrt{\beta})\right] , \tag{4}$$

where $U(f) > 0$ is a *temperature-independent* potential with support in $\{f(x)\} < 0$. The total potential $\beta V(f) + U(f)$ is indicated by the full curve in Fig. 1 where we have denoted the depth of the well $\beta V(f)$ by βg_0 and the height of $U(f)$ by g_1.

As the temperature is varied, the depth of the well changes while the height of the barrier remains constant. Exact results for $D=1$ show that for each value of g_1 there is a critical value $(\beta g_0)^*$ where the model (4) exhibits a pinning transition. This line of critical points persists down to infinitesimal potential strenghts and, furthermore, the universal features of the transition—the critical exponents, for example—are the same along the entire critical line.[9]

It is therefore sufficient to analyze the critical behavior of a model defined by the Lagrangian

$$\mathcal{L} = \int d^D x \, \{\tfrac{1}{2}(\nabla f)^2 + g_1 \exp[-\lambda_0(f/\sqrt{\beta}+\sigma)^2] - \beta g_0 \exp[-\lambda_0(f/\sqrt{\beta}-\sigma)^2]\} , \tag{5}$$

where g_0 and g_1 are infinitesimal. The potentials $\beta V(f)$ and $U(f)$ have been taken here to be Gaussians. This choice proves to be the most convenient; however, the choice of other short-range potentials leads to similar results.

It is well known that for $D=(d-1)>2$ the interface between coexisting phases in an Ising ferromagnet is always sharp.[10] For $D \leq 2$, however, long-wavelength interface fluctuations have sufficient density to destroy a sharp interface. In a field theory such as (5) this effect manifests itself as infrared divergences in massless graphs that contain closed loops consisting of a single internal line—the so-called tadpole graphs. These divergences appear first in $D=2$.

We are therefore interested in the critical behavior of the Lagrangian (5) for $D \leq 2$. Since g_0 and g_1 are infinitesimal we have employed a renormalization-group (RG) analysis of this model based on a cumulant expansion in g_0 and g_1.

In performing a perturbation expansion with (5), we are faced with the problem of regularizing the previously mentioned infrared divergences for $D \leq 2$. There are various methods of dealing with this problem. Here we have found the following procedure to be the most convenient.[11] We add a mass term $\frac{1}{2}m_0^2 f^2$ to the free part of the Lagrangian

$$\mathcal{L}_0 = \int d^D x \, [\tfrac{1}{2}(\nabla f)^2 + \tfrac{1}{2}m_0^2 f^2] ,$$

and subtract it from the perturbative part

$$\mathcal{L}_1 = \int d^D x \, \{g_1 \exp[-\lambda_0(f/\sqrt{\beta}+\sigma)^2] - \beta g_0 \exp[-\lambda_0(f/\sqrt{\beta}-\sigma)^2] - \tfrac{1}{2}m_0^2 f^2\} .$$

The mass $m_0 \neq 0$ is arbitrary and the total Lagrangian remains unchanged by this procedure.

A convenient ultraviolet (uv) regularization is introduced by defining the free propagator in real space to be[12]

$$G(m_0,x) = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ipy}}{p^2+m_0^2}\bigg|_{y^2=x^2+a^2} ,$$

where a is a short-distance cutoff. This regularization is preferable to the usual sharp momentum cutoff since the required calculations are most easily performed in coordinate space.

Since explicit calculations to second order in $\mathcal{L}_1$ show no uv divergences in momentum-dependent graphs, it suffices to consider the effective potential $U(f)$ here. To first order in $\mathcal{L}_1$ we find

$$U(f) = \frac{1}{2}\int \frac{d^D q}{(2\pi)^D} \ln(m_0^2+q^2) - \frac{1}{2}m_0^2 G_0(m_0) - \frac{\beta g_0}{[1+2\lambda_0 G_0(m_0)/\beta]^{1/2}} \exp\left(-\frac{\lambda_0(f/\sqrt{\beta}-\sigma)^2}{1+2\lambda_0 G_0(m_0)/\beta}\right) + \frac{g_1}{[1+2\lambda_0 G_0(m_0)/\beta]^{1/2}} \exp\left(-\frac{\lambda_0(f/\sqrt{\beta}+\sigma)^2}{1+2\lambda_0 G_0(m_0)/\beta}\right) , \tag{6}$$

where $G_0(m_0) = G(m_0, x=0)$.

The interaction part of the effective potential can be renormalized by introducing one renormalization function Z. Introducing an arbitrary momentum scale κ, the renormalization of this term is uniquely defined by introducing dimensionless renormalized coupling constants λ, u_0, and u_1 via

$$\lambda_0 = \kappa^\epsilon Z \lambda , \tag{7a}$$

$$g_i = \kappa^D Z^{1/2} u_i \quad (i=0,1) , \tag{7b}$$

where the Z function is given by

$$Z^{-1}=1-2\kappa^{\epsilon}\lambda G_0(\kappa)/\beta \ , \tag{7c}$$

and $\epsilon=2-D$. m_0 and σ require no renormalization, and, furthermore, no wave-function renormalization is necessary. The above procedure is equivalent to normal ordering the Lagrangian (resumming all tadpole graphs) at mass κ. Luckily, (5) exhibits none of the pathologies of the sine-Gordon theory,[12] and this procedure appears to renormalize the theory completely. The relations (7) go beyond the loop expansion and are exact to the extent that they renormalize the theory to arbitrary order. Since the first two terms in the effective potential do not play a role in the following discussion, the subtractions necessary to renormalize them will not be discussed here.

The normal RG procedure may now be applied to establish the model's critical behavior.[13] Relations (7) imply that the coupling constants flow according to

$$\rho\frac{d\lambda}{d\rho}=-\left(\epsilon-\frac{b\lambda}{\beta}\right)\lambda \ , \tag{8a}$$

$$\rho\frac{du_i}{d\rho}=-\left(D-\frac{\frac{1}{2}b\lambda}{\beta}\right)u_i \ , \tag{8b}$$

as the length scale is changed by a factor ρ. The constant b in (8) is given by

$$b=2^{1-D}\pi^{-D/2}\epsilon\Gamma(\epsilon/2) \ .$$

For $\epsilon\to 0$,

$$b \longrightarrow 1/\pi \ .$$

Equations (8) are easily solved in closed form. From (8a) we find

$$\lambda(\rho)=\begin{cases}\dfrac{\lambda(1)}{1-[\lambda(1)/\pi\beta]\ln\rho} \ , & D=2 \\ \dfrac{\epsilon\lambda(1)}{\lambda(1)(1-\rho^{\epsilon})b/\beta+\epsilon\rho^{\epsilon}} \ , & D<2 \ .\end{cases}$$

Thus for $D=2$ the theory is asymptotically free, while for $D<2$ there is a stable fixed point at $\lambda^*=\beta\epsilon/b$. Finally, from (8b) we obtain

$$u_i(\rho)=u_i(1)\rho^{-2+(3/2)\epsilon}[\lambda(\rho)/\lambda(1)]^{1/2} \ .$$

In order to utilize these results we need to consider the renormalized effective potential $U_R(f)$. The equilibrium value of the order parameter $v=\langle f\rangle$ is given by the solution of

$$\Gamma_R^{(1)}(k=0)=\left.\frac{\partial U_R}{\partial f}\right|_{f=v}=0 \ , \tag{9a}$$

and since there is no wave-function renormalization, the inverse square of the correlation length ξ^{-2} is given by

$$\xi^{-2}=\Gamma_R^{(2)}(k=0)|_{f=v}=\left.\frac{\partial^2 U_R}{\partial f^2}\right|_{f=v} \ . \tag{9b}$$

Higher-order vertices are given by the corresponding derivatives of U_R.

Since both $\Gamma_R^{(1)}$ and $\Gamma_R^{(2)}$ obey homogeneous RG equations, the infrared behavior of v and ξ^{-2} may be determined by solving (9) simultaneously in the infrared ($\rho\to 0$) limit. Specifically, we have employed the standard matching procedure of integrating out to a $\rho=\rho^*$ such that $\xi^{-2}(\rho^*)=\kappa^2\rho^{*2}$.[13]

Applying this approach we find a continuous phase transition at a critical temperature $T_p=1/k_B\beta_p$ given by the solution of

$$\beta_p u_0\exp(4\sigma\sqrt{2\pi\beta_p})=u_1$$

for $D=2$. For $1<D<2$ we find a first-order transition, while for $\frac{2}{3}<D\leq 1$ we again find a continuous transition at temperature $\beta_p u_0=u_1$. Denoting the reduced temperature by τ we find that ξ diverges at T_p as

$$\xi=\begin{cases}\tau^{-3/8}\exp(c/\tau) \ , & D=2 \\ \tau^{-2/(4-3\epsilon)} \ , & \frac{2}{3}<D\leq 1\end{cases} \tag{10}$$

for $\tau>0$ ($T<T_p$), where c is a nonuniversal constant. For $D=2$ the correlation length has an essential singularity, while for $\frac{2}{3}<D\leq 1$ we find

$$\nu=2/(4-3\epsilon) \ .$$

Furthermore, the mean distance v of the interface from the pinning potential diverges as

$$v\sim 1/\tau$$

for both $\frac{2}{3}<D\leq 1$ and $D=2$ as the pinning transition is approached from below. Above T_p, $v^{-1}=\xi^{-1}=0$ so that the interface is delocalized and rough.

These results indicate a lower critical dimension of $D=\frac{2}{3}$ (bulk dimension $d=\frac{5}{3}$) for the transition. For $D<\frac{2}{3}$, the coupling constants u_i have stable fixed points at $u_i=0$. Short-range pinning potentials of the type considered here are therefore irrelevant for $D<\frac{2}{3}$, and we conclude that there is no longer a pinning transition in this case. The domain wall is delocalized and rough at all temperatures.

In the first-order region, the size of the jump Δv^{-1} of v^{-1} from the pinned to the unpinned phase decreases as $D\to 1+$. Letting $\delta=D-1$, Δv^{-1} goes to zero as

$$\Delta v^{-1}\sim e^{-c/\delta}$$

as $\delta\to 0+$, where c is a positive constant. As a function of D, therefore, the transition in $D = 1$ is at a multicritical point.

These methods may be extended to study the free energy F and the various interface cumulants $\langle f^n \rangle_c$. The singular part of the free energy is found to scale as

$$F \sim \xi^{-D}$$

for $D = 2$ and $\frac{2}{3} < D \leq 1$. For $n \geq 3$ the cumulants behave as

$$\langle f^n \rangle_c \sim \begin{cases} \text{const} , & D = 2 \\ \tau^{-n} , & \frac{2}{3} < D \leq 1 \end{cases} ,$$

while for $n = 2$ we find

$$\langle f^2 \rangle_c \sim \begin{cases} \ln \xi , & D = 2 \\ \xi^{\epsilon} , & \frac{2}{3} < D \leq 1 \end{cases} ,$$

where the temperature dependence of the correlation length ξ is given by (10). These results are in complete agreement with known exact results for $D = 1$.

Until now we have ignored any effects of the crystal lattice. For $d = 3$, however, the lattice plays an essential role in determining the behavior of the domain wall. The results obtained here are, in fact, only applicable for relatively strong pinning potentials for which T_p lies above T_R, the roughening temperature of the three-dimensional Ising ferromagnet. For situations in which $T_p > T_R$ the lattice is irrelevant and a transition of the type of described here occurs. For weaker pinning potentials the effect of the lattice may not be ignored and the situation is more complicated. This case will be discussed elsewhere.

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[1]D. B. Abraham, Phys. Rev. Lett. 44, 1165 (1980).
[2]T. W. Burkhardt, J. Phys. A 14, L63 (1981).
[3]J. T. Chalker, J. Phys. A 14, 2431 (1981).
[4]S. T. Chui and J. D. Weeks, Phys. Rev. B 23, 2438 (1981).
[5]J. M. J. van Leeuwen and H. J. Hilhorst, Physica (Utrecht) 107A, 319 (1981).
[6]D. M. Kroll, Z. Phys. B 41, 345 (1981).
[7]M. Vallade and J. Lajzerowicz, J. Phys. (Paris) 42, 1505 (1981).
[8]The only other work we know of concerning the $D \neq 1$ pinning problem are the mean-field theory of T. W. Burkhardt and V. R. Vieira, J. Phys. A 14, L223 (1981) and the pinning-modified Coulomb-gas duality relation derived by J. T. Chalker (unpublished).
[9]D. M. Kroll (unpublished).
[10]See, for example, J. M. Kosterlitz, J. Phys. C 10, 3753 (1977) or D. Jasnow and J. Rudnick, Phys. Rev. Lett. 41, 698 (1978).
[11]P. Minnhagen, A. Rosengren, and G. Grinstein, Phys. Rev. B 18, 1356 (1978).
[12]D. J. Amit, Y. Y. Goldschmidt, and G. Grinstein, J. Phys. A 13, 585 (1980).
[13]D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena* (McGraw-Hill, New York, 1978).