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On the Theory of Turbulence: A non Eulerian Renormalized Expansion

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A non Eulerian framework for a renormalized theory of isotropic homogeneous steady state turbulence at high Reynold's numbers is developed. By construction it is invariant under random Galilei transformations. A direct interaction factorization is free of infrared singularities and yields Kolmogorov scaling for the static as well as for the dynamic correlation and response functions.

1. Introduction

Renormalized perturbation expansions have turned out to be most valuable in many fields of physics. In a statistical theory of turbulence based on an Eulerian description of the fluid and the Navier-Stokes equation the simplest nontrivial truncation leads to the direct interaction approximation (DIA) of Kraichnan [1]. In contrast to an order by order expansion the renormalized expansion in this case leads to spurious effects for the energy transfer at small scales by convection at large scales [2]. A manifestation of this is the violation of Galilei invariance, and as a consequence the DIA gives an incorrect exponent for the energy spectrum in the inertial subrange. It is quite doubtful whether this deficiency can be cured by vertex renormalizations or similar means.

A way out of this dilemma has been proposed again by Kraichnan [3, 4]. If the Eulerian framework is replaced by a generalized Lagrangian description, the theory can be made invariant under Galilei transformations in each order of a renormalized expansion. The resulting DIA-equations are, however, quite involved and additional simplifications are necessary to bring them into a tractable form. The energy spectrum obtained from such a Lagrangian version of the DIA shows Kolmogorov 41 scaling [5] in contrast to the Eulerian DIA. Corrections due to intermittency [6] are not contained in a DIA and the Lagrangian DIA therefore behaves as expected.

In the present paper we propose an alternative non Eulerian framework which is Galilei invariant and free of the spurious effects due to convection at large scales typical for an Eulerian renormalized expansion. The DIA in this new picture yields Kolmogorov 41 scaling like in the Lagrangian DIA. In contrast to this latter approach the equations are only slightly more complex than those of the Eulerian DIA.

The situation to be studied is isotropic, homogeneous, steady state turbulence driven by Gaussian correlated fluctuating forces. The spectrum of the forces is assumed to vanish outside a narrow band around some wavenumber $K_0 \sim L_0^{-1}$, where L_0 plays the role of the external length scale. Furthermore, no white noise spectrum is assumed contrary to most investigations on steady state turbulence driven by random forces. This more general form appears to be required by the fact that the large scale motions, simulated by the fluctuating forces, are correlated over times much longer than the characteristic time-scales of the small scale motions which are investigated. This of course means that one has to abandon a Fokker-Planck description. Instead a path integral formulation of the Wyld-Martin-Siggia-Rose formalisms [7] is used.

In order to give some motivation for the following let us repeat some well-known heuristic arguments based on naive dimensional analysis. We focus on the time

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dependent correlation function

$$C_{\alpha\beta}(\mathbf{x}-\mathbf{x}', t-t') = \langle u_\alpha(\mathbf{x}, t)\, u_\beta(\mathbf{x}', t')\rangle \tag{1.1}$$

where $u_\alpha(\mathbf{x}, t)$ are the cartesian components of the Eulerian velocity field. For isotropic turbulence its Fourier transform

$$C_{\alpha\beta}(\mathbf{k}, t) = \int d^d x\, C_{\alpha\beta}(\mathbf{x}, t) \exp(-i\mathbf{k}\cdot\mathbf{x}) = P_{\alpha\beta}(\mathbf{k})\, C(k, t) \tag{1.2}$$

is the product of a scalar function $C(k,t)$ and a transverse projection operator $P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2$. The energy spectrum (in d dimensions) is

$$E(k) = \frac{d-1}{(4\pi)^{d/2}\,\Gamma(d/2)}\, k^{d-1}\, C(k, 0). \tag{1.3}$$

For sufficiently high Reynold's numbers a power law behaviour of $E(k)$ is expected for k in the inertial subrange, $K_0 \ll k \ll \kappa_d$. $1/\kappa_d$ is the dissipation length. Eddies of this size are damped by viscosity. In the inertial subrange viscous damping is negligible and under the assumption that K_0 also plays no role [5] the energy spectrum has the well-known Kolmogorov form

$$E(k) = c\, \varepsilon^{2/3}\, k^{-5/3} \tag{1.4}$$

where ε is the total energy dissipated per unite time and volume and c is Kolmogorov's constant.

It is tempting to try the same naive dimensional analysis for the dynamic correlation function. Assuming

$$C(k, t) = C(k, 0)\, f(t/\tau(k)) \tag{1.5}$$

the resulting characteristic time is

$$\tau(k) \sim \varepsilon^{-1/3}\, k^{-2/3}. \tag{1.6}$$

This is certainly not correct even if intermittency corrections are neglected because of sweeping effects. This means that in a turbulent flow small scale structures are convected by the large scale flow. In order to make this more clear we assume that the velocity field is split into two parts

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t) + \hat{\mathbf{u}}(\mathbf{x} - \mathbf{y}(t), t) \tag{1.7}$$

where $\mathbf{v}(\mathbf{x}, t)$ describes the large scale motions. It is supposed to contain Fourier components out of the band $K_0 < k < \mu$ only. $\hat{\mathbf{u}}(\mathbf{x}-\mathbf{y}(t), t)$ describes the small scale motion. The frame of reference is, however, not the Eulerian but rather a coordinate system advected with $\mathbf{v}(\mathbf{x}, t)$. A crude estimate can now be given if the spacial dependence of $\mathbf{v}$ is neglected,

$$\mathbf{y}(t) = \int_0^t d\tau\, \mathbf{v}(\tau) \tag{1.8}$$

and if $\mathbf{v}(\mathbf{x}, t)$ is replaced by an appropriate ensemble over $\mathbf{v}(t)$. This yields for the Fourier transform of the correlation function with $k > \mu$

$$C(k, t-t') = \int \mathscr{D}\{\mathbf{v}(\tau)\}\, \mathscr{P}(\mathbf{v}(\tau)) \cdot \hat{C}(k, t-t') \exp[-i\mathbf{k}\cdot\{\mathbf{y}(t) - \mathbf{y}(t')\}] \tag{1.9}$$

where an average has been taken over the process $\mathbf{v}(\tau)$ with a weigth $\mathscr{P}(\mathbf{v}(\tau))$. $\hat{C}(k,t)$ is the correlation function of $\hat{\mathbf{u}}$. Assuming a Gaussian distribution and neglecting the time dependence of $\mathbf{v}(t)$ one finds

$$C(k, t) = \exp\{-\langle v^2\rangle k^2 t^2 / 2d\}\, \hat{C}(k, t). \tag{1.10}$$

Since $\mathbf{v}$ represents the large scale motions a reasonable choice is

$$\langle v^2\rangle = \int_{K_0}^{\mu} dk\, E(k) \simeq \tfrac{3}{2} c\, \varepsilon^{2/3} \{K_0^{-2/3} - \mu^{-2/3}\}. \tag{1.11}$$

Assuming that the characteristic time in $\hat{C}(k,t)$ is given by its naive dimensions, Eq. (1.6), the time dependence in $C(k,t)$ is ruled by the first factor in (1.10) having a characteristic time, the sweeping time

$$\tau_s(k) \sim \varepsilon^{-1/3}\, K_0^{1/3}\, k^{-1} \tag{1.12}$$

since $\tau_s(k) \ll \tau(k)$ for $k \gg K_0$.

This crude estimate demonstrates clearly the well-known fact that an Eulerian description is not suited as the basis of a theory of turbulence at high Reynold's numbers or, equivalently, in the limit cutoff $K_0 \to 0$, since the whole dynamics is dominated by sweeping effects. In any finite order of a renormalized perturbation theory the intrinsic cutoff dependence of the dynamics appears to carry over into the statics and gives rise to incorrect exponents, for instance $E(k) \sim k^{-3/2}$ in the Eulerian DIA.

In our present investigation we use (1.9) and its generalization to higher order correlation and response functions together with (1.8) as a definition of new non Eulerian velocity fields $\hat{\mathbf{u}}(\mathbf{x}, t)$. Starting from the Eulerian equations of motion the theory is reformulated for the "randomly advected fields" $\hat{\mathbf{u}}(\mathbf{x}, t)$. With an appropriate choice of the probability functional $\mathscr{P}(\mathbf{v}(\tau))$ defining the transformation from $\mathbf{u}(\mathbf{x}, t)$ to $\hat{\mathbf{u}}(\mathbf{x}, t)$ the theory is free of the spurious sweeping effects and it is invariant under Galilei transformations. A DIA yields Kolmogorov 41 scaling for the energy spectrum and the characteristic timescale in $\hat{C}(k, t)$ behaves as expected from naive dimensional analysis, Eq. (1.6).

Galilei invariant DIA equations have been proposed recently by Kuznetsov and L'vov [8] using related arguments. Their scheme differs, however, in two essential points. We obtain their equations by choosing a time independent advecting field determined by

(1.11) with $\mu \to \infty$. In order to find correlation and response functions which vanish for long times it appears, however, to be necessary to choose $\mathbf{v}(\tau)$ correlated over finite times and to choose μ of the order of k.

The present paper is organized as follows. In Sect. 2 a path integral formulation [7] of homogeneous isotropic steady state turbulence in an Eulerian description is reviewed and the propagator renormalization and DIA are discussed. Section 3 deals with the transformation to the randomly advected field formulation and the DIA in this scheme is investigated in Sect. 4. The inertial range behaviour of this DIA is discussed in Sect. 5 and some conclusions are given in Sect. 6.

2. Eulerian Framework

We study the small scale motions of stationary isotropic non rotational homogeneous fully developed turbulence in an incompressible fluid. The steady state is maintained by external random forces. The forces replace the large scale motions, for instance the largest eddies created behind a grid. If the grid size is L_0 they have a typical wavenumber $K_0 = 1/L_0$ and accordingly the external forces should be chosen to have Fourier components only around this wavenumber. The timescale for those eddies is $\tau \sim \varepsilon^{-1/3} K_0^{-2/3}$ and therefore much longer than the typical timescales for the small scale motions. The temporal behaviour of the fluctuating forces should be chosen accordingly.

We are interested in averages over an ensemble of fluctuating forces assuming a Gaussian distribution with zero mean. The ensemble is then specified by the second moment of the forces $\mathbf{f}(\mathbf{x}, t)$

$$\langle f_\alpha(\mathbf{x}, t) f_\beta(\mathbf{x}', t')\rangle = P_{\alpha\beta}(\nabla)\, \gamma_0(|\mathbf{x}-\mathbf{x}'|, t-t'). \tag{2.1}$$

According to the above discussion the Fourier components of $\gamma_0(x, t)$ have to vanish outside a region around the wavenumber K_0 and a frequency $\Omega_0 \sim \varepsilon^{1/3} K_0^{2/3}$.

The fluid is described by the Navier-Stokes equation for an incompressible fluid with the random forces added. A second external forces field $\tilde{\varphi}(\mathbf{x}, t)$ is introduced which serves later to define response functions. The resulting equation reads

$$\partial_t u_\alpha(\mathbf{x}, t) + P_{\alpha\beta\gamma}(\nabla)\, u_\gamma(\mathbf{x}, t)\, u_\beta(\mathbf{x}, t) - \nu \Delta \mathbf{u}_\alpha(\mathbf{x}, t)$$
$$= f_\alpha(\mathbf{x}, t) + \tilde{\varphi}_\alpha(\mathbf{x}, t) \tag{2.2}$$

where ν is the kinematic viscosity, $P_{\alpha\beta\gamma}(\nabla) = P_{\alpha\beta}(\nabla)\,\nabla_\gamma$ and a summation over repeated indices is implied.

Following standard techniques a generating functional

$$G(\varphi, \tilde{\varphi}) = \langle \exp \textstyle\int d^d x\, dt\, \varphi_\alpha(\mathbf{x}, t)\, u_\alpha(\mathbf{x}, t)\rangle \tag{2.3}$$

is defined. The average is taken over the ensemble of fluctuating forces and $\mathbf{u}(\mathbf{x}, t)$ is considered as a functional of f and $\tilde{\varphi}$ and is a solution of (2.2). Actually $\mathbf{u}$ depends also on initial conditions at some t_0 which become irrelevant in the limit $t_0 \to -\infty$. This functional generates correlation- and response functions

$$G^{(n,m)}_{\alpha_1\ldots\alpha_n\beta_1\ldots\beta_m}(\mathbf{x}_1, t_1 \ldots \mathbf{x}_n, t_n; \mathbf{y}_1, s_1 \ldots \mathbf{y}_m, s_m)$$
$$= \frac{\delta^m}{\delta\tilde{\varphi}_{\beta_1}(\mathbf{y}_1, s_1) \ldots \delta\tilde{\varphi}_{\beta_m}(y_m, s_m)}$$
$$\cdot \langle u_{\alpha_1}(\mathbf{x}_1, t_1) \ldots u_{\alpha_n}(\mathbf{x}_n, t_n)\rangle$$
$$= \frac{\delta^{m+n}}{\delta\tilde{\varphi}_{\beta_1}(\mathbf{y}_1, s_1) \ldots \delta\varphi_{\alpha_1}(\mathbf{x}_1, t_1) \ldots} G(\varphi, \tilde{\varphi})\bigg|_{\varphi=\tilde{\varphi}=0}. \tag{2.4}$$

Of special interest are the correlation function

$$C_{\alpha\alpha'}(\mathbf{x}-\mathbf{x}', t-t') = G^{(2,0)}_{\alpha\alpha'}(\mathbf{x}, t, \mathbf{x}', t')$$
$$= P_{\alpha\alpha'}(\nabla)\, C(|\mathbf{x}-\mathbf{x}'|, t-t') \tag{2.5}$$

and the response function

$$R_{\alpha\beta}(\mathbf{x}-\mathbf{y}, t-s) = G^{(1,1)}_{\alpha\beta}(\mathbf{x}, t; \mathbf{y}, s)$$
$$= P_{\alpha\beta}(\nabla)\, R(|\mathbf{x}-\mathbf{y}|, t-s). \tag{2.6}$$

The representation of $C_{\alpha\alpha'}$ and $R_{\alpha\beta}$ by scalar functions C and R is of course a consequence of the assumed isotropy and and absence of rotation [9].

The Navier-Stokes equation (2.2) and the definition (2.3) yield the following equation of motion

$$\partial_t\, \delta\varphi_\alpha(\mathbf{x}, t)\, G(\varphi, \tilde{\varphi})$$
$$= \{-P_{\alpha\beta\gamma}(\nabla)\, \delta\varphi_\gamma(\mathbf{x}, t)\, \delta\varphi_\beta(\mathbf{x}, t) - \nu\Delta\,\delta\varphi_\alpha(\mathbf{x}, t)$$
$$+ \tilde{\varphi}_\alpha(\mathbf{x}, t) + \textstyle\int P_{\alpha\beta}(\nabla)\, \gamma_0(|\mathbf{x}-\mathbf{y}|, t-s)\, \delta\tilde{\varphi}_\beta(\mathbf{y}, t)\}\, G(\varphi, \tilde{\varphi}) \tag{2.7}$$

where the notation $\delta\varphi_\alpha(\mathbf{x}, t) = \delta/\delta\varphi_\alpha(\mathbf{x}, t)$ is used and $\int$ denotes integration over repeated space and time variables. The last term in (2.7) originates from the $f_\alpha(\mathbf{x}, t)$ in (2.2) realizing that $\mathbf{f}$ is Gaussian distributed and that $\mathbf{u}$ depends on $\mathbf{f} + \tilde{\varphi}$ only.

A formal solution of the equation of motion (2.7) is

$$G(\varphi, \tilde{\varphi}) = \exp\{\tfrac{1}{2}\textstyle\int \delta\tilde{\varphi}_\alpha(\mathbf{x}, t)\, \delta\tilde{\varphi}_\beta(\mathbf{x}', t')$$
$$\cdot\, P_{\alpha\beta}(\nabla)\, \gamma'(|\mathbf{x}-\mathbf{x}'|, t-t')$$
$$- \textstyle\int \delta\tilde{\varphi}_\alpha(\mathbf{x}, t)\, \delta\varphi_\beta(\mathbf{x}', t')\, P_{\alpha\beta}(\nabla)\, \eta'(|\mathbf{x}-\mathbf{x}'|, t-t')$$
$$- \textstyle\int \delta\tilde{\varphi}_\alpha(\mathbf{x}, t)\, P_{\alpha\beta\gamma}(\nabla)\, \delta\varphi_\beta(\mathbf{x}, t)\, \delta\varphi_\gamma(\mathbf{x}, t)\}$$
$$\cdot\, \exp\{\tfrac{1}{2}\textstyle\int \varphi_\alpha(\mathbf{x}, t)\, \varphi_\beta(\mathbf{x}', t')\, P_{\alpha\beta}(\nabla)\, \bar{C}(|\mathbf{x}-\mathbf{x}'|, t-t')$$
$$+ \textstyle\int \varphi_\alpha(\mathbf{x}, t)\, \tilde{\varphi}_\beta(\mathbf{x}', t')\, P_{\alpha\beta}(\nabla)\, \bar{R}(|\mathbf{x}-\mathbf{x}'|, t-t')\}. \tag{2.8}$$

This solution superficially depends on two arbitrary functions $\gamma(x,t)$ and $\eta(x,t)$, where

$$\gamma'(x,t)=\gamma_0(x,t)-\gamma(x,t),$$
$$\eta'(x,t)=-\nu\Delta\delta(\mathbf{x})\,\delta(t)-\eta(x,t). \tag{2.9}$$

The response propagator in (2.8) obeys

$$\partial_t\bar{R}(|\mathbf{x}-\mathbf{x}'|,t-t')=-\int\eta(|\mathbf{x}-\mathbf{y}|,t-s)\,\bar{R}(|\mathbf{y}-\mathbf{x}'|,s-t')+\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t') \tag{2.10}$$

with causal boundary condition $\bar{R}(x,t)=0$ for $t\leqq 0$ implying $\eta(x,t)=0$ for $t<0$. The correlation propagator is

$$\bar{C}(|\mathbf{x}-\mathbf{x}'|,t-t')=\int\bar{R}(|\mathbf{x}-\mathbf{y}|,t-s)\cdot\gamma(|\mathbf{y}-\mathbf{y}'|,s-s')\,\bar{R}(|\mathbf{x}'-\mathbf{y}'|,t'-s'). \tag{2.11}$$

Actually $G(\varphi,\tilde{\varphi})$ has to be independent on the arbitrary functions $\gamma(x,t)$ and $\eta(x,t)$ which is most easily seen if one shows that (2.8) is a solution of (2.7) for $\gamma=\eta=0$ and verifies that $\delta G(\varphi,\tilde{\varphi})/\delta\gamma=\delta G(\varphi,\tilde{\varphi})/\delta\eta=0$ for any γ and η.

Propagator renormalization consists in choosing γ and η such that $\bar{R}=R$ and $\bar{C}=C$ and determining the counter terms involving γ' and η' in (2.9) accordingly. Similar techniques can be employed in order to perform vertex renormalizations [7]. Such a renormalization has actually been discussed in the context of an incompressible fluid stirred by fluctuating forces with a white noise spectrum acting at all wavenumbers [10], a problem not directly related to three-dimensional turbulent flow at high Reynold's numbers. At present we investigate a propagator renormalized expansion in the bare interaction represented by the third integral in the first exponent of (2.8).

The DIA consists in retaining the first nontrivial contributions to the selfenergies γ and η only. These are of second order in the interaction. The resulting expressions are

$$\gamma(x,t)=\gamma_0(x,t)-\frac{1}{d-1}P_{\alpha\beta\gamma}(\nabla)\cdot\{P_{\alpha\delta\varepsilon}(\nabla)+P_{\alpha\varepsilon\delta}(\nabla)\}\,C_{\beta\delta}(\mathbf{x},t)\,C_{\gamma\varepsilon}(\mathbf{x},t),$$
$$\eta(x,t)=-\nu\Delta\,\delta(\mathbf{x})\,\delta(t)-\frac{1}{d-1}\{P_{\alpha\beta\gamma}(\nabla)+P_{\alpha\gamma\beta}(\nabla)\}\cdot C_{\beta\delta}(\mathbf{x},t)\{P_{\alpha\delta\varepsilon}(\nabla)+P_{\alpha\varepsilon\delta}(\nabla)\}\,R_{\gamma\varepsilon}(\mathbf{x},t) \tag{2.12}$$

where $P_{\alpha\alpha}(\nabla)=d-1$ has been used. Equations(2.10–12) together with the definitions (2.5, 6) of the scalar parts form the well-known DIA equations. The corresponding equations for the Fourier transforms are

$$R(k,\omega)=\int\exp(-i\mathbf{k}\cdot\mathbf{x}+i\omega t)\,R(x,t)=\{-i\omega+\eta(k,\omega)\}^{-1},$$
$$C(k,\omega)=R(k,\omega)\,\gamma(k,\omega)\,R^*(k,\omega) \tag{2.13}$$

and

$$\gamma(k,t)=\gamma_0(k,t)+k^2\int_\Delta dp\,dq\,F(k,p,q)\,a(k,p,q)\,C(p,t)\,C(q,t),$$
$$\eta(k,t)=\nu k^2+k^2\int_\Delta dp\,dq\,F(k,p,q)\,b(k,p,q)\,R(p,t)\,C(q,t). \tag{2.14}$$

The coefficients $a(k,p,q)$ and $b(k,p,q)$ arise from the Fourier transforms of the differential operators in (2.12) and (2.5, 6) [1, 11]. Since $a(k,p,q)+a(k,q,p)=b(k,p,q)+b(k,q,p)$ the coefficient $a(k,p,q)$ might be replaced by $b(k,p,q)$ in (2.14). In d-dimensions the latter can be written as

$$b(k,p,q)=\tfrac{1}{4}c(k,p,q)\left\{\frac{k^2}{q^2}-\frac{1}{d-1}\frac{k^2+p^2+q^2}{p^2}\right\},$$
$$c(k,p,q)=(2k^2p^2+2k^2q^2+2p^2q^2-k^4-p^4-q^4). \tag{2.15}$$

The d-dimensional integrations in momentum space have been converted into integrations over triangles. The boundaries of the remaining two-dimensional integrals are $p+q>k>|p-q|$ with $p>0$ and $q>0$. The factor

$$F(k,p,q)=\frac{k^{d-4}\,p\,q\,\{c(k,p,q)\}^{(d-3)/2}}{4^{d-2}\,\pi^{(d+1)/2}\,\Gamma((d-1)/2)} \tag{2.16}$$

originates from this transformation and the $d-2$ angular integrations.

The origin of the spurious sweeping effects in the Eulerian DIA mentioned in the introduction can now easily be seen. The factor $F(k,p,q)\,b(k,p,q)$ behaves as q^{d-2} in the limit $q\to 0$. Assuming Kolmogorov scaling for $C(q)$ for $q\geqq K_0$ and using K_0 as a lower cutoff, the integrals in (2.14) diverge as $K_0^{-2/3}$ in the limit $K_0\to 0$. The region $p\to 0$ is not dangerous since $F(k,p,q)\,b(k,p,q)$ behaves as p^d in this limit.

3. Randomly Advected Field Formalisms

The need for a reformulation of the theory in terms of non Eulerian velocities has been stated in the introduction. The present formulation is based on randomly advected fields. This means the statistical properties of the actual velocity field $\mathbf{u}(\mathbf{x},t)$ are expressed by the statistical properties of a velocity field $\hat{\mathbf{u}}(\mathbf{x},t)$ which is advected by a spacially constant but time dependent velocity field $\mathbf{v}(t)$ and averaged over a given distribution $\mathscr{P}(v)$ of $\mathbf{v}(t)$.

Let us define the generating functional $\hat{G}(\varphi,\tilde{\varphi})$ for the response and correlation functions of the randomly advected fields

$$G(\{\varphi(\mathbf{x},t)\},\{\tilde{\varphi}(\mathbf{x},t)\})=\int \mathscr{D}\{v(\tau)\}\,\mathscr{P}(v)$$
$$\cdot\,\hat{G}(\{\varphi(\mathbf{x}+\mathbf{y}(t),t)\},\{\tilde{\varphi}(\mathbf{x}+\mathbf{y}(t),t)\}). \tag{3.1}$$

This definition implies (1.9). As stated in (1.8)

$$\mathbf{y}(t)=\int_0^t d\tau\,\mathbf{v}(\tau). \tag{3.2}$$

We assume that we can find the inverse of the above transformation. For a Gaussian $\mathscr{P}(v)$ the explicit form is given later. Then

$$\hat{G}(\{\varphi(\mathbf{x},t)\},\{\tilde{\varphi}(\mathbf{x},t)\})=\int \mathscr{D}\{v(\tau)\}\,\mathscr{P}^{-1}(v)$$
$$\cdot\,G(\{\varphi(\mathbf{x}+\mathbf{y}(t),t)\},\{\tilde{\varphi}(\mathbf{x}+\mathbf{y}(t),t)\}). \tag{3.3}$$

The original equation of motion (2.7) yields for $\hat{G}$

$$\partial_t\,\delta\varphi_\alpha(\mathbf{x},t)\,\hat{G}(\varphi,\tilde{\varphi})=\int \mathscr{D}\{v(\tau)\}\,\mathscr{P}^{-1}(v)[\tilde{\varphi}_\alpha(\mathbf{x},t)$$
$$+P_{\alpha\beta\gamma}(\nabla)\{v_\gamma(t)-\delta\varphi_\gamma(\mathbf{x},t)\}\,\delta\varphi_\beta(\mathbf{x},t)+\nu\Delta\,\delta\varphi_\alpha(x,t)$$
$$+\int P_{\alpha\beta}(\nabla)\,\gamma_0(|\mathbf{x}-\mathbf{x}'-\mathbf{y}(t)+\mathbf{y}'(t')|,t-t')$$
$$\cdot\,\delta\tilde{\varphi}_\beta(\mathbf{x}',t')]\,\hat{G}(\varphi,\tilde{\varphi}). \tag{3.4}$$

The formal solution corresponding to (2.8) is

$$\hat{G}(\varphi,\tilde{\varphi})=\int \mathscr{D}\{v(\tau)\}\,\mathscr{P}^{-1}(v)$$
$$\cdot\,\exp[\tfrac{1}{2}\int \delta\tilde{\varphi}_\alpha(\mathbf{x},t)\,\delta\tilde{\varphi}_\beta(\mathbf{x}',t')$$
$$\cdot\,P_{\alpha\beta}(\nabla)\,\hat{\gamma}'(\mathbf{x}-\mathbf{x}',\mathbf{y}(t)-\mathbf{y}(t'),t-t')$$
$$-\int \delta\tilde{\varphi}_\alpha(\mathbf{x},t)\,\delta\varphi_\beta(\mathbf{x}',t')\,P_{\alpha\beta}(\nabla)\,\hat{\eta}'(|\mathbf{x}-\mathbf{x}'|,t-t')$$
$$+\int \delta\tilde{\varphi}_\alpha(\mathbf{x},t)\,P_{\alpha\beta\gamma}(\nabla)\{v_\gamma(t)-\delta\varphi_\gamma(\mathbf{x},t)\}\,\delta\varphi_\beta(\mathbf{x},t)]$$
$$\cdot\,\exp[\tfrac{1}{2}\int \varphi_\alpha(\mathbf{x},t)\,\varphi_\beta(\mathbf{x}',t')\,P_{\alpha\beta}(\nabla)\,\hat{C}(|\mathbf{x}-\mathbf{x}'|,t-t')$$
$$+\int \varphi_\alpha(\mathbf{x},t)\,\tilde{\varphi}_\beta(\mathbf{x}',t')\,P_{\alpha\beta}(\nabla)\,\hat{R}(|\mathbf{x}-\mathbf{x}'|,t-t')] \tag{3.5}$$

where

$$\hat{\gamma}'(\mathbf{x}-\mathbf{x}',\mathbf{y}(t)-\mathbf{y}(t'),t-t')$$
$$=\gamma_0(|\mathbf{x}-\mathbf{x}'+\mathbf{y}(t)-\mathbf{y}(t')|,t-t')-\hat{\gamma}(|\mathbf{x}-\mathbf{x}'|,t-t'),$$
$$\hat{\eta}'(|\mathbf{x}-\mathbf{x}'|,t-t')=\nu\Delta\,\delta(\mathbf{x}-\mathbf{x}')-\hat{\eta}(|\mathbf{x}-\mathbf{x}'|,t-t'). \tag{3.6}$$

The propagators $\hat{C}$ and $\hat{R}$ are solutions of (2.10) and (2.11), respectively, with γ and η replaced by $\hat{\gamma}$ and $\hat{\eta}$.

The transformation (3.1) has the following group property. Assume we have two probability functionals $\mathscr{P}(v;\mu)$ and $\mathscr{P}(v;\mu')$. Defining

$$\mathscr{P}(v;\mu,\mu')=\int \mathscr{D}\{v'(\tau)\}\,\mathscr{P}^{-1}(v-v';\mu)\,\mathscr{P}(v';\mu') \tag{3.7}$$

we can write

$$\hat{G}(\{\varphi(\mathbf{x},t)\},\{\tilde{\varphi}(\mathbf{x},t)\};\mu)=\int \mathscr{D}\{v(\tau)\}\,\mathscr{P}(v;\mu,\mu')$$
$$\cdot\,\hat{G}(\{\varphi(\mathbf{x}-\mathbf{y}(t),t)\},\{\tilde{\varphi}(x-y(t),t)\};\mu') \tag{3.8}$$

where $\hat{G}(\varphi,\tilde{\varphi};\mu)$ is given by (3.1) with $\mathscr{P}(v;\mu)$. The similarity between (3.8) and (3.1) suggests the notation

$$G(\varphi,\tilde{\varphi})=\hat{G}(\varphi,\tilde{\varphi};0),\qquad \mathscr{P}(v;\mu)=\mathscr{P}(v;0,\mu)$$

and

$$\mathscr{P}^{-1}(v;\mu)=\mathscr{P}(v;\mu,0).$$

For the following discussion of the DIA in the present formulation it is sufficient to consider Gaussian correlated advecting fields only. The distribution functional is

$$\mathscr{P}(v;\mu,\mu')=\mathscr{N}(\mu,\mu')$$
$$\cdot\,\exp\left\{-\tfrac{1}{2}\int dt\,dt'\,\frac{v_\alpha(t)\,v_\alpha(t)}{V(t-t';\mu,\mu')}\right\} \tag{3.9}$$

where $\mathscr{N}(\mu,\mu')$ is a normalization factor and

$$V(t-t';\mu,\mu')=d^{-1}\int \mathscr{D}\{v(\tau)\}\,v_\alpha(t)\,v_\alpha(t')\,\mathscr{P}(v;\mu,\mu') \tag{3.10}$$

is the second moment of the advecting velocities. The inverse transformation is determined by

$$\mathscr{P}^{-1}(v;\mu,\mu')=\mathscr{P}(v;\mu',\mu)$$
$$=\mathscr{N}(\mu,\mu')\exp\left\{\tfrac{1}{2}\int dt\,dt'\,\frac{v_\alpha(t)\,v_\alpha(t)}{V(t-t';\mu,\mu')}\right\} \tag{3.11}$$

but the integration in (3.3) now runs over imaginary advecting velocities $\mathbf{v}(\tau)$. Obviously $V(t-t';\mu',\mu)=-V(t-t';\mu,\mu')$.

For Gaussian correlated advecting velocities the displacements $\mathbf{y}(t)$ are also Gaussian correlated with second moment

$$Y(t-t';\mu,\mu')=d^{-1}\int \mathscr{D}\{v(\tau)\}[y_\alpha(t)-y_\alpha(t')]^2\,\mathscr{P}(v;\mu,\mu')$$
$$=2\int_0^{t-t'} d\tau\,(t-t'-\tau)\,V(\tau;\mu,\mu'). \tag{3.12}$$

The asymptotic expressions for small and large times are

$$Y(t;\mu,\mu')\xrightarrow[t\to 0]{} V(0;\mu,\mu')\,t^2,$$
$$Y(t;\mu,\mu')\xrightarrow[t\to\infty]{}\int_0^\infty d\tau\,V(\tau;\mu,\mu')\,t-\int_0^\infty d\tau\,\tau\,V(\tau;\mu,\mu') \tag{3.13}$$

provided $V(\tau;\mu,\mu')$ vanishes rapidly enough for $\tau\to\infty$.

The effect of the transformation on the correlation propagator $\hat{C}$ is most easily expressed in Fourier space where

$$\hat{C}(k,t;\mu)=\exp\{\tfrac{1}{2}\,Y(t;\mu,\mu')\,k^2\}\,\hat{C}(k,t;\mu'). \tag{3.14}$$

$\hat{R}$, $\hat{\gamma}$ and $\hat{\eta}$ transform accordingly.

4. Direct Interaction Approximation

As in the Eulerian framework a propagator renormalized perturbation expansion is obtained by expansion of the first exponential in (3.5). Its exponent contains the advecting field linearly in its last term and also in $\hat{\gamma}'$ or γ_0. For reasons given later we can neglect the dependence of $\hat{\gamma}$ or γ_0 in (3.5) on the advecting field if we are interested in the small scale behaviour only. Statements concerning the large scale motions can not be made within the framework of a stationary turbulence driven by fluctuating forces because those motions depend on details of the mechanism injecting energy into the system and on boundary conditions.

The DIA involves again expansion of the first exponent in (3.5) up to second order in $P_{\alpha\beta\gamma}(V)$ and the corresponding determination of the counterterms $\hat{\gamma}'$ and $\hat{\eta}'$. The resulting expressions for the selfenergies are similar to those of the Eulerian DIA (2.14) but an extra term arises from the average over the advecting field

$$\begin{aligned}\hat{\gamma}(k,t;\mu)&=\gamma_0(k,t)+k^2\,\hat{C}(k,t;\mu)\,V(t;\mu,0)\\&+k^2\int_{\Delta} dp\,dq\,F(k,p,q)\,b(k,p,q)\,\hat{C}(p,t;\mu)\,\hat{C}(q,t;\mu),\\ \hat{\eta}(k,t;\mu)&=\nu k^2\,\delta(t)+k^2\,\hat{R}(k,t;\mu)\,V(t;\mu,0)\\&+k^2\int_{\Delta} dp\,dq\,F(k,p,q)\,b(k,p,q)\,\hat{R}(p,t;\mu)\,\hat{C}(q,t;\mu).\end{aligned} \tag{4.1}$$

The integrals in this expression are, of course, still dominated by the contributions arising from $q\sim K_0$ and diverge for $K_0\to 0$. We have, however, not yet specified the second moment of the advecting field and we can choose this quantity such that this divergency is cancelled by a corresponding divergency in $V(t;\mu,\mu')$ in the limit $K_0\to 0$. A possible choice is

$$\begin{aligned}V(t;\mu,\mu')&=-\int_{\Delta\mu,\mu'} dp\,dq\,F(k,p,q)\\&\cdot b(k,p,q)\,\hat{C}(q,t;\alpha q)\end{aligned} \tag{4.2}$$

where $\int_{\Delta\mu,\mu'}$ indicates integration over p and q satisfying the inequalities $p>0$, $\mu>q>\mu'\geqq 0$, $p+q>k>|p-q|$ assuming $\mu>\mu'$. For $\mu<\mu'$ the choice $V(t;\mu,\mu')=-V(t;\mu',\mu)$ is consistent with (3.11).

In the introduction we argued that the advecting field has to simulate the advection of the small scale structures by the large scale motions. If we are interested in motions with wavevector $k\gg K_0$ it is reasonable to count the motions with wavevector $q<\alpha k$ as large scale motions with $1>\alpha>0$. This means an appropriate choice for μ in (4.1) is $\mu(k)=\alpha k$. From $\hat{\gamma}(k,t;\alpha k)$ and $\hat{\eta}(k,t;\alpha k)$ the functions $\hat{C}(k,t;\alpha k)$ and $\hat{R}(k,t;\alpha k)$ are obtained from (2.19). In the integrals of (4.1) we need, however, $\hat{C}(q,t;\mu)$ and $\hat{R}(q,t;\mu)$ with $\mu=\alpha k\neq\alpha q$. On the other hand, (3.14) allows to calculate these quantities from values for $\mu=\alpha q$ and the special choice (4.2) also involves these values of μ only.

The resulting DIA now contains functions with $\mu(\kappa)=\alpha\kappa$ only and we can drop this variable again. The selfenergies are

$$\begin{aligned}\hat{\gamma}(k,t)&=\gamma_0(k,t)+k^2\,\hat{C}(k,t)\,W(k,t)\\&+k^2\int_{\Delta} dp\,dq\,F(k,p,q)\,b(k,p,q)\\&\cdot\{\hat{C}(p,t)\exp X(k,p,q,t)-\hat{C}(k,t)\}\,\hat{C}(q,t),\\ \hat{\eta}(k,t)&=\nu k^2\,\delta(t)+k^2\,\hat{R}(k,t)\,W(k,t)\\&+k^2\int_{\Delta} dp\,dq\,F(k,p,q)\,b(k,p,q)\\&\cdot\{\hat{R}(p,t)\exp X(k,p,q,t)-\hat{R}(k,t)\}\,\hat{C}(q,t)\end{aligned} \tag{4.3}$$

with

$$X(k,p,q,t)=-\tfrac{1}{2}\{Y(t;\alpha k,\alpha q)\,q^2+Y(t;\alpha k,\alpha p)\,p^2\} \tag{4.4}$$

and

$$W(k,t)=\int_{\Delta\infty,\alpha k} dp\,dq\,F(k,p,q)\,b(k,p,q)\,\hat{C}(q,t). \tag{4.5}$$

(4.2) and (3.12) yields

$$\begin{aligned}Y(t;\alpha k,\alpha q)&=-2\int_0^t d\tau\,(t-\tau)\int_{\Delta\alpha k,\alpha q} d\bar{p}\,d\bar{q}\\&\cdot F(k,\bar{p},\bar{q})\,b(k,\bar{p},\bar{q})\,\hat{C}(\bar{q},\tau)\end{aligned} \tag{4.6}$$

for $k>q$ and $Y(t;\alpha k,\alpha q)=-Y(t;\alpha q,\alpha k)$ for $k<q$. The set of equations is closed by

$$\begin{aligned}\hat{R}(k,\omega)&=\{-i\omega+\hat{\eta}(k,\omega)\}^{-1},\\ \hat{C}(k,\omega)&=\hat{R}(k,\omega)\,\hat{\gamma}(k,\omega)\,\hat{R}^*(k,\omega).\end{aligned} \tag{4.7}$$

The above equations are almost identical to the original Eulerian DIA equations (2.13–14) but the selfenergies (4.3) contain extraterms which remove the infrared singularities in the limit $K_0\to 0$. As discussed in the next section Kolmogorov scaling solutions are therefore expected.

Let us come back to the discussion at the beginning of this section where we had argued that the influence of the advection on γ_0 may be neglected. We are concerned with $\gamma_0(k,t)\neq 0$ for $k\approx K_0$ only. This means $\gamma_0(k,t)$ does not contribute to $\hat{\gamma}(k,t)$ unless $k\approx K_0$. In this case the strength of the advecting field vanishes according to (4.2) because $\mu\approx\mu'\approx K_0$.

5. Scaling Solutions in the Inertial Subrange

The above equations can be brought into dimensionless form using k^{-1} as lengthscale and

$$\tau(k)=\varepsilon^{-1/3}\,k^{-2/3} \tag{5.1}$$

as timescale. The resulting dimensionless functions can depend only on dimensionless variables like $s=t/\tau(k)$ or K_0/k or κ_d/k, where $\kappa_d=\varepsilon^{1/4}\,v^{-3/4}$ is the inverse of the dissipation length. In the inertial subrange, this is for $K_0 \ll k \ll \kappa_d$, a scaling form of the solutions of (4.3–7) is expected, for instance

$$C(k,t)=Z\,\varepsilon^{2/3}\,k^{-d-\frac{2}{3}+\xi}\,\tilde{C}(Z_t\,\varepsilon^{1/3}\,k^{\frac{2}{3}+\zeta}\,t). \tag{5.2}$$

Kolmogorov 41 scaling or naive dimensional analysis yields $\xi=0$ and $\zeta=0$. The Z-factors have to be homogeneous functions of K_0 and κ_d of degree $-\xi$ and $-\zeta$, respectively. If, on the other hand, finite solutions of the above equations are found in the limit $K_0\to 0$ and $\kappa_d\to\infty$ the anomalous dimensions ξ and ζ have to be zero and Kolmogorov scaling is established.

In order to see whether this is the case we have to study the infrared and the ultraviolet behaviour of the triangle-integrations in (4.3–6). As in the Eulerian DIA the limit $\kappa_d\to\infty$ or $v\to 0$ is not problematic. The limit $K_0\to 0$, however, causes divergencies in the Eulerian DIA arising from small q values. This is not the case in the present scheme as we shall see. Writing $p=k+p'$ the integration extends over $-q<p'<q$ and we have to investigate the behaviour of the bracket in the integrand of (4.3). For $t=0$ we can simply expand around $p=k$ because $X(k,p,q,0)=0$ and

$$\hat{C}(p,0)-\hat{C}(k,0)\sim p'\sim q. \tag{5.3}$$

This is sufficient to make the integral convergent in the limit $K_0\to 0$.

At finite t we have to study the behaviour of $X(k, k+p', q, t)$. The second term in (4.4) is proportional to p' as is easily seen from (4.6). Using the estimate

$$\hat{C}(q,t)<C\,q^{-d-\frac{2}{3}} \tag{5.4}$$

in (4.6) we find

$$-\tfrac{1}{2}Y(t;\alpha k,\alpha q)<\tfrac{1}{2}C\,t^2\int\limits_{\Delta\alpha k,\alpha q} d\bar{p}\,d\bar{q}\,F(k,\bar{p},\bar{q})\cdot b(k,\bar{p},\bar{q})\,\bar{q}^{-d-\frac{2}{3}}\xrightarrow[q\to 0]{} C'\,t^2\,q^{-2/3} \tag{5.5}$$

where C and C' are finite constants. This means $X(k, k+p', q, t)$ vanishes at least proportional to q in the limit $q\to 0$ and the expression in the bracket of (4.3) is proportional to q for all finite times. This means $\hat{\gamma}(k,t)$ and $\hat{\eta}(k,t)$ are finite in the limit $K_0\to 0$.

There is a nontrivial point concerning the long time behaviour of the present scheme. This is due to the fact that $Y(t;\,p,q)$ given by (4.6) increases asymptotically linear in time. Since this quantity enters in the exponent of the exponentials in (4.3) the response and correlation functions have to decay at least exponentially in time with an exponent large enough such that $\hat{\gamma}(k,t)\to 0$ and $\hat{\eta}(k,t)\to 0$ for $t\to\infty$.

To make this point more precise we assume that $\hat{C}(k,t)$ and $\hat{R}(k,t)$ decay exponentially in time. In other words we neglect the ω-dependence of $\hat{\gamma}(k,\omega)$ and $\hat{\eta}(k,\omega)$ in (4.7). This yields

$$\hat{R}(k,\omega)=\{-i\omega+\hat{\eta}(k)\},$$

$$\hat{C}(k,\omega)=\hat{R}(k,\omega)\,\hat{\gamma}(k)\,\hat{R}^*(k,\omega) \tag{5.6}$$

and

$$\hat{R}(k,t)=\exp\{-\hat{\eta}(k)t\}\,\Theta(t),$$

$$\hat{C}(k,t)=\frac{\hat{\gamma}(k)}{2\hat{\eta}(k)}\exp\{-\hat{\eta}(k)|t|\}. \tag{5.7}$$

This is certainly not correct at small times but might be sufficient for an estimate of the long time behaviour. The above choice is motivated by the assumption that $\hat{R}(k,\omega)$ can be approximated by a single pole and therefore

$$\hat{\gamma}(k)=\hat{\gamma}(k,-i\hat{\eta}(k))$$

$$=\int_0^\infty dt[\exp\{\hat{\eta}(k)t\}+\exp\{-\hat{\eta}(k)t\}]\,\hat{\gamma}(k,t),$$

$$\hat{\eta}(k)=\hat{\eta}(k,-i\hat{\eta}(k))$$

$$=\int_0^\infty dt\exp\{\hat{\eta}(k)t\}\,\hat{\eta}(k,t). \tag{5.8}$$

Writing (5.6) we have implicitly assumed

$$\left.\frac{\partial\hat{\eta}(k,\omega)}{\partial\omega}\right|_{\omega=-i\hat{\eta}(k)}\ll 1.$$

A naive scaling assumption yields

$$\hat{\eta}(k)=\varepsilon^{1/3}\,k^{2/3}\,\tilde{\eta},$$

$$\frac{\hat{\gamma}(k)}{2\hat{\eta}(k)}=\varepsilon^{2/3}\,k^{-d-2/3}\,\tilde{C} \tag{5.9}$$

where $\tilde{\eta}$ and $\tilde{C}$ are dimensionless constants, the latter is proportional to Kolmogorov's constant.

Writing the above equations in terms of dimensionless variables and inserting the special timedependence one obtains after some manipulations from the second equation in (5.8)

$$\frac{\tilde{\eta}^2}{\tilde{C}}=\int_\alpha^\infty dy\,B(y)\,y^{-7/2}+\int_\Delta dx\,dy\,B(x,y)\cdot\left\{\frac{y^{-5/3}\exp\{-\tilde{X}_1(x)-\tilde{X}_1(y)\}}{x^{2/3}+y^{2/3}-\tilde{X}_0(x)-\tilde{X}_0(y)-1}-y^{-7/2}\right\} \tag{5.10}$$

where

$$B(x,y)=y^{-d+1}F(1,x,y)\,b(1,x,y),$$
$$B(y)=\int_{\Delta} dx\,B(x,y). \tag{5.11}$$

In order to perform the time-integrations the asymptotic form (3.13) for late times has been used, resulting in

$$\tilde{X}_0(x)=\frac{\tilde{C}}{\tilde{\eta}^2}x^2\int_{\alpha x}^{\alpha} dy\,B(y)\,y^{-7/3},$$
$$X_1(x)=\frac{\tilde{C}}{\tilde{\eta}^2}x^2\int_{\alpha x}^{\alpha} dy\,B(y)\,y^{-3}. \tag{5.12}$$

An estimate of these quantities is obtained if $B(y)$ is replaced by its limit

$$B(0)=\{(4\pi)^{d/2}\,\Gamma(\tfrac{1}{2}d+1)\}. \tag{5.13}$$

This results in

$$\tilde{X}_0(x)=\frac{3B(0)\,\tilde{C}}{4\alpha^{4/3}\tilde{\eta}^2}\{x^{2/3}-x^2\},$$
$$\tilde{X}_1(x)=\frac{B(0)\,\tilde{C}}{2\alpha^2\tilde{\eta}^2}\{1-x^2\}. \tag{5.14}$$

The denominator in the second term of (5.10) originates from the time integral of an exponential. This integral is finite only if the expression in the denominator is positive. With the estimate (5.14) it is easily seen that the minimum of this denominator inside the integration range of x and y is at $x=y=1/2$. As as consequence

$$\frac{B(0)\,\tilde{C}}{\alpha^{4/3}\tilde{\eta}^2}<\frac{4(1-2^{-1/3})}{3(1-2^{-4/3})}\simeq 0.456. \tag{5.15}$$

Another requirement is, of course, that the integral in the second term of (5.10) is not infrared divergent. This is not automatically fulfilled despite the discussion at the beginning of this section because the limit $K_0/k\to 0$ was discussed only for finite times, whereas now an integration over all positive times has been performed. This requirement yields with (5.14) and keeping $1-y<x<1+y$ in mind

$$\exp\left\{-\frac{B(0)\,\tilde{C}}{2\alpha^2\tilde{\eta}^2}\right\}=1-\frac{3B(0)\,\tilde{C}}{4\alpha^{4/3}\tilde{\eta}^2}$$

or

$$\alpha=\left\{\frac{-2\beta}{3\ln(1-\beta)}\right\}^{3/2} \quad \text{with} \quad \beta=\frac{3B(0)\,\tilde{C}}{4\alpha^{4/3}\tilde{\eta}^2}. \tag{5.16}$$

For $0<\beta<1$ real values of α are found and $\alpha<(2/3)^{3/2}\simeq 0.816$. Eq. (5.15) yields together with (5.16) a lower bound $\alpha>0.353$.

The first equation in (5.8) results in an expression similar to (5.10) which is also free of singularities if (5.15) and (5.16) is fulfilled. Both equations depend on the combination $\tilde{C}/\tilde{\eta}^2$ only. This means that they can be fulfilled for special values of α only. This is not surprising since the same exponent for the time dependence of the response- and correlation function (5.7) has been assumed.

From the fact that the above equations are free of singularities we can, however, conclude that Kolmogorov scaling solutions with more general time dependence are likely to exist. A conclusive answer will be obtained from numerical calculations which have not yet been performed.

6. Concluding Remarks

In this paper, we have proposed a non Eulerian framework for a statistical theory of turbulence based on randomly advected velocity fields. The transformation from the Eulerian to the randomly advected velocity fields involves a distribution functional of an advecting field. As a special choice a Gaussian distribution has been investigated and this allows to eliminate the leading infrared singularity. A direct interaction approximation is then free of divergencies and consequently exhibits Kolmogorov 41 scaling in the inertial subrange for the statics as well as for the dynamics. The Eulerian correlation and response functions are identical to those of the randomly advected fields if all time arguments are identical. In general, they are obtained by an average over the advecting fields. As a consequence their characteristic timescale is the cutoff-dependent sweeping time.

In higher orders of a renormalized perturbation theory additional infrared singularities are expected. Some of them can be removed by allowing more general non Gaussian distributions for the advecting fields. Other singularities are expected to persist giving rise to anomalous dimensions, for instance intermittency corrections to the exponent of the energy spectrum. Important questions are, of course, whether the number of such singularities is finite, whether the theory is renormalizable and whether a marginal dimension [12] exists. In an attempt to answer those questions the presently proposed non Eulerian DIA might play a similar role as does the Gaussian model for phase transitions.

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