Z. Phys. B – Condensed Matter 55, 335–343 (1984)

Condensed Matter
Zeitschrift für Physik B

Long-Range Correlations at Depinning Transitions. I

R. Lipowsky
Sektion Physik der Universität, München, Federal Republic of Germany

Received February 20, 1984

Interface delocalization or depinning transitions such as wetting or surface induced disorder are considered. At these transitions, the correlation length $\xi_{\parallel}$ for transverse correlations parallel to the surface diverges. These correlations are studied in the framework of Landau theory. It is shown that $\xi_{\parallel} \propto |t|^{-1/2}$ at all types of transitions for systems with short-range forces where t measures the distance from bulk coexistence.

1. Introduction

At a first-order phase transition, several thermodynamic bulk phases coexist. Recently, it has been realized that a surface can induce critical effects at such transitions [1–3]. These effects are due to the interplay of the surface with an interface which separates two (almost) coexisting phases (see Fig. 1). The interface may delocalize or depin from the surface in a continuous way i.e. its distance $\hat{\ell}$ from the surface may diverge. As $\hat{\ell}$ goes to infinity, long-range correlations build up parallel to the surface which are characterized by the diverging correlation length $\xi_{\parallel}$ (see Fig. 1).

Such depinning transitions have first been studied near the coexistence curve of a liquid and a gas. In this context, they are known as wetting or drying [2]. If a symmetry is broken at the first-order bulk transition, a disordered phase and several ordered phases can be distinguished. In this case, the depinning transitions have been called surface induced order and disorder (SIO and SID) [3].

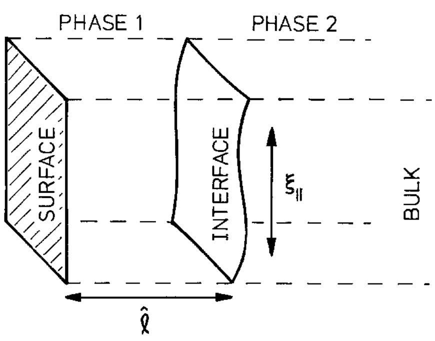

Fig. 1. The interface separates phase 1 near the surface from phase 2 in the bulk. Both $\hat{\ell}$ and $\xi_{\parallel}$ diverge at the various depinning transitions. Note that $\xi_{\parallel}$ and $\hat{\ell}$ have not been drawn to scale: $\xi_{\parallel}$ diverges more strongly than $\hat{\ell}$

The critical behaviour near the surface can be characterized by surface critical exponents. These exponents are related by scaling laws. The number of *independent* surface exponents leads to a classification of surface criticality. In the simplest case, there is only one such exponent. I will refer to this case as *protocritical* depinning. (The expression 'protocritical' has been used before in a different context namely in order to denote the bulk critical behaviour of the Yang-Lee-edge singularity [4].) For the coexistence of a gas and a liquid, the protocritical transition is usually called *complete* wetting. From an experimental point of view, this type of surface criticality is the most important one since it can occur for the widest range of microscopic surface couplings.

In the following, the behaviour of the correlation length $\xi_{\parallel}$ (see Fig. 1) is systematically studied in the framework of Landau theory. In this paper, only short-range surface fields will be considered. However, at all types of depinning transitions, longrange surface fields are a relevant perturbation which can be characterized by another independent surface exponent. This will be shown in the subsequent paper.

The results described below should be applicable to the coexistence of two liquids or of a liquid with a gas. If the coexisting phases have an underlying lattice structure, this can be a relevant perturbation which truncates the divergence of $\xi_{\parallel}$. Such effects are not included here.

This paper is organized as follows. Section 2 and Appendix A contain a review of previous definitions and results which are used in the rest of the paper. In Sect. 3, the eigenvalue problem related to the Gaussian fluctuations is derived. These fluctuations contain a soft mode (Sect. 4). The soft mode energy $E_0=(\xi_{\|})^{-2}$ is determined in Sect. 4.1 for the critical and multicritical depinning transitions, and in Sect. 4.2 for the protocritical (or complete) transition. Finally, the singular part of the mean-field correlation function is discussed in Sect. 5 and Appendix B. In particular, it is shown that Landau theory already predicts a diverging interface width $\xi_{\perp}$ for $d \leqq 3$. Some of the results have been reported in [3, 5].

For the protocritical (or complete) case, long-range correlations have been predicted previously from the van der Waals theory for fluids [6, 7]. However, the singular behaviour of $\xi_{\|}$ which is determined below has not been obtained before. Apart from Ref. 5, previous work [8–11] on the critical transition has been restricted to the behaviour of $\xi_{\|}$ *at* coexistence. In contrast, I will emphasize the *approach to* coexistence which is governed by a different surface critical exponent.

2. Landau-Ginzburg Models for First-Order Phase Transitions

The Landau-Ginzburg (LG) models studied in this paper are of the generic form [1, 3]

$$F\{\phi\}=(A_0^*)^2 \int d^{d-1}\rho \int_0^\infty dz[\tfrac{1}{2}(\nabla\phi)^2+f(\phi)+\delta(z) f_1(\phi)] \tag{1}$$

for the scalar order parameter field $A_0^*\phi$. ϕ is dimensionless, and A_0^* is the order parameter amplitude of the ordered bulk phase at coexistence. In (1), z denotes the coordinate perpendicular to the surface at $z=0$, and $\boldsymbol{\rho}$ are the $(d-1)$ coordinates parallel to it. The bulk potential $f(\phi)$ is [3]

$$f(\phi)=(\xi_d^*)^{-2}\tfrac{1}{2}\phi^2[1+t-2\phi^p+\phi^{2p}] \tag{2}$$

ξ_d^* is the correlation length of the disordered bulk phase at coexistence. Both A_0^* and ξ_d^* are input parameters for the above model, and have to be taken from experiments, computer simulations, or microscopic theories. *The dimensionless variable t in* (2) *measures the distance from bulk coexistence.* If a quantity is considered at coexistence ($t=0$), this is indicated by a star (*).

The surface term $f_1(\phi)$ in (1) is taken to be

$$f_1(\phi)=-h_1\phi+\tfrac{1}{2}a_1\phi^2-(2+p)^{-1}b_1\phi^{2+p} +(2+2p)^{-1}c_1\phi^{2+2p} \tag{3}$$

h_1 is a symmetry breaking field at the surface, and a_1 is related to the ratio of surface to bulk couplings. Note that all surface parameters h_1, a_1, b_1, and c_1 have the dimension of an inverse length since ϕ is dimensionless.

The type of transition which occurs in the semi-infinite system as bulk coexistence ($t=0$) is approached depends crucially on the values of the Landau parameters in (3). The phase diagram for $t=0$ and $b_1=c_1=0$ as obtained in Landau theory is displayed in Fig. 2 [3]. A superscript $(-)$ indicates a surface induced *disorder* (SID) transition which occurs for $t\to 0^-$. A $(+)$ indicates surface induced *order* (SIO) which occurs for $t\to 0^+$. The protocritical, critical, and tricritical transitions are denoted by $(P^{\pm})$, $(C^{\pm})$, and $(T^{\pm})$ respectively. The (h_1, a_1)-coordinates for the various transitions are given in [3].

The order parameter profiles $M(z):=\langle\phi\rangle$ are displayed in Fig. 3a, b [3]. Note that all profiles have a point of inflection at $z=:\hat{\ell}$. $\hat{\ell}$ is implicitly defined by $M(z=\hat{\ell})=\hat{M}$ with $\hat{M}$ from (A.3). At $(P^{\pm})$, $(C^{\pm})$, and $(T^{\pm})$, the length scale $\hat{\ell}$ diverges. At the protocritical transitions $(P^{\pm})$, Landau theory yields

$$\hat{\ell}=\xi_b^*\ln(1/|t|) \tag{4}$$

with $\xi_b^*=\xi_d^*$ at (P^+) and $\xi_b^*=\xi_0^*$ at (P^+). At the critical and tricritical transitions $(C^{\pm})$ and $(T^{\pm})$, one finds

$$\hat{\ell}=\xi_b^*\ln(1/\delta M_1) \tag{5}$$

with $\delta M_1=M_1$ at SID and $\delta M_1=1-M_1$ at SIO. At $(C^{\pm})$ and $(T^{\pm})$, δM_1 goes continuously to zero as discussed in Appendix A. The different SID and SIO transitions are distinguished by their number of relevant scaling fields. At $(P^{\pm})$, t is the only relevant field. In contrast, there are two and three such fields at $(C^{\pm})$ and $(T^{\pm})$ respectively. For $b_1, c_1 \neq 0$, a multicritical SID transition (Q^-) occurs with four relevant scaling fields (see Appendix A).

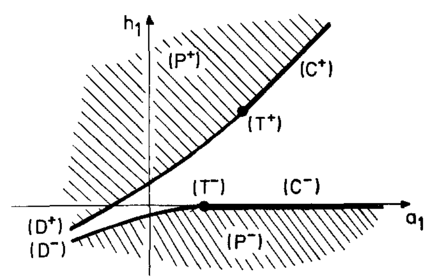

Fig. 2. Phase boundaries inside the bulk coexistence surface ($t=0$). SID and SIO transitions are indicated by $(-)$ and $(+)$ since they occur as the coexistence surface is approached from $t<0$ and $t>0$, respectively [3]

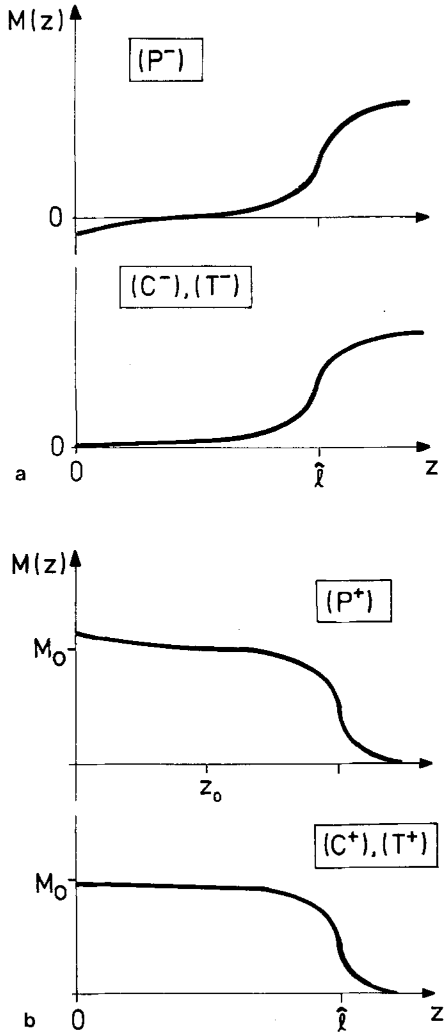

Fig. 3a and b. Order parameter profiles $M(z)$: **a** at the SID transitions (P^-), (C^-), and (T^-); **b** at the SIO transitions (P^+), (C^+), and (T^+). The interface position is denoted by $z=\hat{\ell}$ in each case [3]

For $p=1$, the model (1)-(3) is equivalent to the LG-model discussed in the context of wetting. The bulk variable t is equivalent to the deviation $\delta\mu$ of the bulk chemical potential from its value at coexistence [12]. The SIO (and SID) transitions (P), (C), and (T) in Fig. 1 correspond to complete, critical and tricritical wetting (and drying).

3. Gaussian Fluctuations

Within Landau theory, the correlation function can be expressed by the normal modes of the Gaussian fluctuations (see Sect. 5). These fluctuations are obtained if

$$\phi(\rho, z)=M(z)+\eta(\rho, z) \tag{6}$$

with the Landau profile $M(z)$ is inserted into (1), and the resulting functional is expanded up to second order in η. In terms of the Fourier components

$$\eta(\mathbf{k}, z):=\int d^{d-1}\rho\, e^{-i\mathbf{k}\cdot\boldsymbol{\rho}}\,\eta(\rho, z) \tag{7}$$

one obtains

$$F\{\phi\}=F\{M\}+\frac{1}{2}\int\frac{d^{k-1}k}{(2\pi)^{d-1}}\int_0^\infty dz\,\eta^*(\mathbf{k},z)\,\hat{\mathbb{O}}\,\eta(\mathbf{k},z) + O(\eta^3) \tag{8}$$

with the differential operator

$$\hat{\mathbb{O}}=k^2+\hat{\mathbb{D}}+\delta(z)\,\hat{\mathbb{D}}_1 \tag{9.a}$$

$$\hat{\mathbb{D}}=-d^2/dz^2+f''(M(z)) \tag{9.b}$$

$$\hat{\mathbb{D}}_1=-d/dz+f_1''(M(z)). \tag{9.c}$$

A prime indicates a derivative with respect to ϕ. In order to determine the normal modes, one has to solve the eigenvalue problem [5]

$$\hat{\mathbb{D}}\,g_n(z)=E_n\,g_n(z) \tag{10.a}$$

with the boundary condition

$$\hat{\mathbb{D}}_1\,g_n(z)|_{z=0}=0 \tag{10.b}$$

(10) is a Schrödinger-type equation for the 1-dimensional motion of a quantum-mechanical particle with the potential

$$Q(z):=f''(M(z))=\ddot{M}(z)/\dot{M}(z) \tag{11}$$

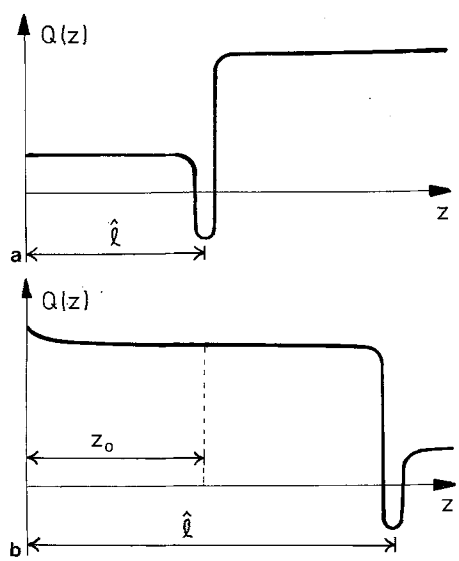

Fig. 4a and b. Schematic shape of the potential $Q(z)$ in the limit $|t|\to 0$ for model (2) with $p=2$: **a** critical and multicritical SID; **b** protocritical SIO where a dot indicates a derivative with respect to z. The potential $Q(z)$ is displayed in Fig. 4a, b for small values of $|t|$. Figure 4a shows the schematic form of $Q(z)$ for the critical and multicritical SID transitions. There is a potential well at $z=\hat{\ell}$. The width of this well is $\simeq \xi_d$ which is finite for $t \to 0^-$. In this limit, one finds

$$(\xi_d^*)^2 Q(z) \sim \begin{cases} 1 & z=0 \\ -p^2/(1+p) & z=\hat{\ell} \\ p^2 & z=\infty. \end{cases} \tag{12}$$

In Fig. 4b, the potential $Q(z)$ for the protocritical SIO transition (P^+) is shown. In this case, the width of the well is $\simeq \xi_0$. For $t \to 0^+$,

$$(\xi_0^*)^2 Q(z) \sim \begin{cases} 1 & z=z_0 \\ -1/(1+p) & z=\hat{\ell} \\ 1/p^2 & z=\infty \end{cases} \tag{13}$$

where z_0 is implicitly defined by $M(z=z_0)=M_0$ (the lower index 0 means 'ordered'). M_0 as given by (A.2) is the order parameter of the ordered phase which is metastable for $t>0$.

4. Soft Mode

If the surface were not present, the ground state $g_0(z)$ would be proportional to $\dot{M}(z)$ and its energy E_0 would be equal to zero [13, 14]. Thus, the differential operator (10a) would possess a zero mode which restores the translational symmetry broken by $M(z)$. The presence of the surface enters through the boundary condition (10b) which lifts the energy E_0 to a positive value. However, E_0 goes continuously to zero as the various transitions are approached. Thus, $g_0(z)$ is a *soft mode.* The asymptotic behaviour of E_0 is derived below by means of upper and lower bounds for the critical and multicritical SID transitions (Sect. 4.1), and for the protocritical SIO transitions (Sect. 4.2). The same methods may be applied to all other cases.

4.1. Critical and Multicritical Transition

From Fig. 3a, one expects that the bound state $g_0(z)$ has a maximum at $z \simeq \hat{\ell}$ and vanishes exponentially for large values of $|z-\hat{\ell}|$. The function $\dot{M}(z)=dM/dz$ has these features. However, $\dot{M}(z)$ does not fulfill the boundary condition (10b). Thus, it can not be used directly in a variational procedure. On the other hand, the trial function

$$\psi(z) := A\, e^{-z/\xi_d} + \dot{M}(z) \tag{14}$$

with

$$A := [f'(M_1) - f_1''(M_1)\, f_1'(M_1)]/[f_1''(M_1) + 1/\xi_d^*] \tag{15}$$

does fulfill the boundary condition (a prime indicates a derivative with respect to ϕ). At the critical and multicritical SID transitions, $\psi(z) \to \dot{M}(z)$ since $A \to 0$. The asymptotic behaviour of A is

$$A \to \begin{cases} (1/\xi_d^* - a_1)\, M_1 & (C^-) \\ \tfrac{1}{2}(p+2)\,(b_1 - 1/\xi_d^*)\, M_1^{p+1} & (T^-) \\ -(p+1)\, c_1\, M_1^{2p+1} & (Q^-) \end{cases} \tag{16}$$

for $h_1=0$. For $h_1 \to 0^+$, A contains the additional term $a_1 h_1/(a_1 + 1/\xi_d^*)$ at all three types of transitions. The asymptotic behaviour of M_1 as a function of the scaling fields is given in Appendix A.

An upper bound $E_0^>$ for E_0 is obtained from the minimax principle. For the trial function (14), one has

$$E_0 \leqq E_0^> := (\psi, \mathbb{D}\psi)/(\psi, \psi). \tag{17}$$

At critical and multicritical SID, the expectation values in (17) have the limiting behaviour

$$(\psi, \mathbb{D}\psi) \to \bar{A} := -A[f'(M_1) + f_1'(M_1)/\xi_d^*] \tag{18}$$

$$(\psi, \psi) \to \sigma^* := (\xi_d^*)^{-1}\, p/(2p+4) \tag{19}$$

($(A_0^*)^2 \sigma^*$ is the surface tension of an interface in the infinite system.) Thus, the upper bound (17) has the leading behaviour

$$E_0^> \to \bar{A}/\sigma^*. \tag{20}$$

A lower bound $E_0^<$ can be obtained via Temple's inequality [15]. If $\bar{E}_0 := (\zeta, \mathbb{D}\zeta)/(\zeta, \zeta) < \Delta < E_1 - E_0$ holds for some number Δ where $E_1 - E_0$ is the gap between the ground state energy and the energy of the first excited state, a lower bound for E_0 is

$$E_0^< := \bar{E}_0 - [(\zeta, \mathbb{D}^2\zeta)/(\zeta, \zeta) - \bar{E}_0^2]/[\Delta - \bar{E}_0] \tag{25}$$

for any trial function ζ. Since the width of the potential well of $Q(z)$ stays finite at the transitions (see Fig. 4a, b), $E_1 - E_0$ stays also finite. If $\zeta = \psi$ is chosen, $\bar{E}_0 = E_0^>$ goes to zero. Thus, the existence of $\Delta > 0$ is guaranteed for this choice.

It can be shown that $(\psi, \mathbb{D}^2\psi)$ is of higher order in M_1 than $E_0^>$. This implies that $E_0^< \to E_0^>$ at the critical and multicritical SID transitions. Thus,

$$E_0 \to E_0^> = \bar{A}/\sigma^* \tag{26}$$

with $\bar{A}$ given by (18). If the limiting behaviour of M_1 as given in Appendix A is used in (26), one finds

$$E_0 \to [Z(p)\xi_d^*]^{-2}\, |t| \tag{27}$$

for $t \to 0^-$ with

$$Z(p) = \begin{cases} [p/(2p+4)]^{1/2} & (C^-) \\ p^{1/2}/(p+2) & (T^-) \\ p^{1/2}/[(2p+4)(p+1)]^{1/2} & (Q^-) \end{cases} \tag{27.a}$$

and

$$E_0 \propto \begin{cases} h_1^2 & (C^-) \\ h_1^{(p+2)/(p+1)} & (T^-) \\ h_1^{(2p+2)/(2p+1)} & (Q^-) \end{cases} \tag{28}$$

for $t=0$ and $h_1 \to 0^+$.

4.2. Protocritical Transition

In this case, the trial function (14) is not useful since δM_1 does *not* vanish at the protocritical transition (P^+). From Fig. 3b, it is obvious that $-\dot{M}(z)$ has a maximum at $z=\hat{\ell}$ and a minimum at $z=z_0$ with $M(z_0)=M_0$. Thus, the ground state $g_0(z)$ will resemble $-\dot{M}(z)$ only for $z \gtrsim z_0$. This motivates the Ansatz

$$\gamma(z) := -\dot{M}(z_\varepsilon)\, e^{-\alpha(z_\varepsilon - z)}\, \theta(z_\varepsilon - z) - \dot{M}(z)\, \theta(z - z_\varepsilon) \tag{29}$$

with

$$z_\varepsilon := z_0 + \varepsilon, \qquad \varepsilon > 0. \tag{30}$$

Note that

$$\gamma(z_\varepsilon) = -\dot{M}(z_\varepsilon) \propto t^{1/2} \tag{31}$$

since

$$\dot{M}(z_0) = -[2f(M_0)]^{1/2} \to -t^{1/2}/\xi_d^* \tag{32.a}$$

and

$$\dot{M}(z_\varepsilon) = \dot{M}(z_0) \cosh(\varepsilon/\xi_0) + O(t). \tag{32.b}$$

The parameter

$$\alpha := \ddot{M}(z_\varepsilon)/\dot{M}(z_\varepsilon) \tag{33}$$

which has the limiting behaviour

$$\alpha \to (\xi_0^*)^{-1}\, \bar{\alpha} := (\xi_0^*)^{-1} \tanh(\varepsilon/\xi_0^*) \tag{33.a}$$

ensures that both $\gamma(z)$ and $\dot{\gamma}(z)$ are continuous at $z = z_\varepsilon$. The parameter $\varepsilon > 0$ can be chosen in such a way that the same trial function yields both an upper and a lower bound for E_0. Such a trial function which satisfies the boundary condition (10.b) at $z=0$ is given by

$$\psi(z) = A\, e^{-z/\xi_0} + \gamma(z) \tag{34}$$

with

$$A = e^{-\alpha z_\varepsilon}[f_1''(M_1)\, \dot{M}(z_\varepsilon) - \ddot{M}(z_\varepsilon)]/[f''(M_1) + 1/\xi_0^*]. \tag{35}$$

At (P^+),

$$A \propto t^{(1+\bar{\alpha})/2} \tag{35.a}$$

with $\bar{\alpha} > 0$ from (33.a). I now proceed in the same way as in Sect. 4.1. An upper bound for E_0 is given by

$$E_0^> = (\psi, \mathbb{D}\psi)/(\psi, \psi) \tag{36}$$

with ψ defined by (34). The asymptotic behaviour of these expectation values is found to be

$$(\psi, \mathbb{D}\psi) \to (1-\bar{\alpha}^2)\, \dot{M}^2(z_\varepsilon)/(2\bar{\alpha}\, \xi_0^*) \tag{37.a}$$

$$(\psi, \psi) \to \sigma^* \tag{37.b}$$

with σ^* as in (19). Since $\dot{M}(z_\varepsilon) \propto t^{1/2}$ from (32.b), (36) leads to

$$E_0^> \propto t \tag{38}$$

at (P^+).

A lower bound $E_0^<$ can again be obtained via Temple's inequality (25) where the trial function ψ defined by (34) is inserted. From (37.a.b) and

$$(\psi, \mathbb{D}^2 \psi) \to (1-\bar{\alpha}^2)^2\, \dot{M}^2(z_\varepsilon)/(2\bar{\alpha}\, \xi_0^{*3})$$

one finds

$$E_0^< \to [1 - \bar{\alpha}^2 - (1-\bar{\alpha}^2)^2/\bar{\Delta}]\, \dot{M}^2(z_\varepsilon)/(2\bar{\alpha}\, \xi_0^*) \tag{39}$$

where $\bar{\Delta} := (\xi_0^*)^2 \Delta$ is the dimensionless gap. The bound (39) is useful if it is positive i.e. if

$$1 - \bar{\alpha}^2 = 1 - \tanh^2(\varepsilon/\xi_0^*) < \bar{\Delta}. \tag{40}$$

Since all estimates used so far in this section are valid for arbitrary ε, one can always find an ε for any $\bar{\Delta} > 0$ such that (40) is fulfilled. For such a choice,

$$E_0^< \propto +t \tag{41}$$

(41) and (38) imply that the soft mode energy vanishes as

$$E_0 \propto t \tag{42}$$

at the protocritical transition (P^+).

It is worth noting that the estimates used in this section depend only on two rather general features of the bulk potential $f(\phi)$, namely on 1, the property that the local minima of $f(\phi)$ have finite curvature i.e. that ξ_d^* and ξ_0^* are finite; and on 2. the fact that $f(M_0) - f(M_d) \propto t$ which implies (32). Thus, (42) holds for any bulk potential $f(\phi)$ with these features.

5. Correlations in Landau Theory

Due to the translational and rotational invariance parallel to the surface, the correlation function C of the order parameter field depends only on $\rho := |\boldsymbol{\rho} - \mathbf{0}|$:

$$C(\rho, z z') = \langle \phi(\boldsymbol{\rho}, z)\, \phi(\mathbf{0}, z') \rangle_c = \langle \eta(\boldsymbol{\rho}, z)\, \eta(\mathbf{0}, z') \rangle. \tag{43}$$

Its Fourier transform with respect to ρ is

$$C(k, z z') = \int d^{d-1} \rho\, e^{-i \mathbf{k} \cdot \boldsymbol{\rho}}\, C(\rho, z z'). \tag{44}$$

Within Landau theory, $C(K, z z')$ is the Green's function of the differential operator (9.a) i.e.

$$[k^2 - d^2/dz^2 + Q(z)]\, C(k, z z') = \delta(z - z'). \tag{45}$$

For $k=0$, the solution of (45) can be found in closed form [16, App. B]:

$$\chi(z z') := C(k=0, z z') = \dot{M}(z)\, \dot{M}(z')\, [1/\bar{B} + v(z_{\min})] \tag{46}$$

with

$$z_{\min} := \min(z, z') \tag{47.a}$$

$$\bar{B} := f_1'(M_1)\, [f_1''(M_1)\, f_1'(M_1) - f'(M_1)] \tag{47.b}$$

$$v(z) := \int_0^z dx\, [\dot{M}(x)]^{-2}. \tag{47.c}$$

The dependence of $\chi(z z')$ on t is discussed in Appendix B.

For $k > 0$, the correlation function (44) may be expressed in terms of the eigenmodes $g_n(z)$ and the corresponding eigenvalues E_n of the differential operator (10.a):

$$C(k, z z') = \sum_n \frac{g_n(z)\, g_n^*(z')}{k^2 + E_n}. \tag{48}$$

The singular part of the correlation function due to the soft mode is

$$\bar{C}(k, z z') = C_{00}(k)\, g_0(z)\, g_0(z') \tag{49.a}$$

with

$$C_{00}(k) = (k^2 + E_0)^{-1} = \int_0^\infty dz \int_0^\infty dz'\, g_0(z)\, g_0(z')\, C(k, z z'). \tag{49.b}$$

The inverse Fourier-transform leads to

$$C_{00}(\rho) = \int \frac{d^{d-1} k}{(2\pi)^{d-1}} \left. \frac{e^{i \mathbf{k} \cdot \mathbf{x}}}{k^2 + E_0} \right|_{x^2 = \bar{\rho}^2} \tag{50}$$

with

$$\bar{\rho} := (\rho^2 + a^2)^{1/2} \tag{50.a}$$

where $1/a$ is a high-momentum cutoff for the continuum model. The integral (50) can be evaluated in closed form:

$$C_{00}(\rho) = \bar{\rho}^{3-d}\, \Omega(\bar{\rho}/\xi_{\parallel}) \tag{51}$$

with

$$\xi_{\parallel} := E_0^{-1/2} \tag{52}$$

$$\Omega(x) := (2\pi)^{(1-d)/2}\, x^{(d-3)/2}\, K_{|d-3|/2}(x) \tag{53}$$

$K_\nu(x)$ is a modified Bessel function [17]. The correlation function $C_{00}(\rho)$ has an intuitive physical meaning which can be understood as follows. First, rewrite the order parameter field as

$$\phi(\boldsymbol{\rho}, z) = M(z) + \sum_n \eta_n(\boldsymbol{\rho})\, g_n(z) = M(z) + \eta_0(\boldsymbol{\rho})\, g_0(z) + \dots. \tag{54}$$

Near the various transitions,

$$g_0(z) \simeq \dot{M}(z)/\sqrt{\sigma^*} \tag{55}$$

for finite values of $|\hat{\ell} - z|$ (see Sect. 4). Thus, the fluctuations $\eta_0(\boldsymbol{\rho})$ may be rewritten as

$$\phi(\boldsymbol{\rho}, z) \simeq M(z) + \eta_0(\boldsymbol{\rho})\, \dot{M}(z)/\sqrt{\sigma^*} \simeq M(z - \zeta(\boldsymbol{\rho})) \tag{56}$$

with $\zeta(\boldsymbol{\rho}) := -\eta_0(\boldsymbol{\rho})/\sqrt{\sigma^*}$ to lowest order in η_0. (56) describes an order parameter profile with the local (i.e. $\boldsymbol{\rho}$-dependent) interface position

$$\ell(\boldsymbol{\rho}) = \hat{\ell} + \zeta(\boldsymbol{\rho}). \tag{57}$$

As a consequence,

$$C_{00}(\boldsymbol{\rho}) = \langle \eta_0(\boldsymbol{\rho})\, \eta_0(\mathbf{0}) \rangle \simeq \langle \ell(\boldsymbol{\rho})\, \ell(\mathbf{0}) \rangle_c \tag{58}$$

i.e. $C_{00}(\boldsymbol{\rho})$ contains the fluctuations of the interface position, and $\xi_{\parallel}$ is the correlation length for these fluctuations.

The asymptotic behaviour of the correlation length can be obtained from the results of Sects. 4.1 and 4.2. At the critical and multicritical SID transitions, (27), (28), and (52) imply

$$\xi_{\parallel} \to \xi_d^*\, Z(p)\, |t|^{-\nu_{\parallel}}, \qquad \nu_{\parallel} = 1/2 \tag{59}$$

for $t \to 0^-$ and $h_1 = 0$, and

$$\xi_{\parallel} \propto \begin{cases} h_1^{-1} & (C^-) \\ h_1^{-(p+2)/(2p+2)} & (T^-) \\ h_1^{-(p+1)/(2p+1)} & (Q^-) \end{cases} \tag{60}$$

for $t=0$ and $h_1 \to 0^+$. At the protocritical SIO transition, (41) and (52) yield

$$\xi_{\|} \propto |t|^{-\nu_{\|}}, \quad \nu_{\|}=1/2. \tag{61}$$

In the context of wetting ($p=1$), t corresponds to the deviation $\delta\mu$ of the bulk chemical potential from its value at coexistence whereas h_1 corresponds to the temperature deviation $|T-T_w|$ from the wetting temperature T_w [12]. Thus,

$$\xi_{\|} \propto |\delta\mu|^{-1/2} \tag{62}$$

at complete, critical, and multicritical wetting. For $\delta\mu=0$,

$$\xi_{\|} \propto |T-T_w|^{-1} \tag{63}$$

at critical wetting, and

$$\xi_{\|} \propto |T-T_w|^{-3/4} \tag{64}$$

at tricritical wetting. (63) has also been obtained in [11] by a different method.

It has been realized before by numerical investigations of the van der Waals theory for fluids that $\xi_{\|}$ diverges at complete wetting [6, 7]. However, the nature of this divergence has not been determined. In Fig. 11 of [7], $\xi_{\|}$ as obtained from the numerical work was plotted as a function of $\ln(\delta\mu)$. Presumably, this was motivated by the logarithmic divergence of $\hat{\ell}$. However, this plot clearly shows that $\xi_{\|}$ does not diverge logarithmically. On the other hand, if one takes these numerical data and plots $\ln(\xi_{\|})$ as a function of $\ln(\delta\mu)$, one finds a straight line with a slope $\simeq -1/2$. Thus, the numerical investigations of the van der Waals theory yield $\nu=1/2$ in accordance with (62).

The behaviour of $C_{00}(\rho)$ as a function of ρ naturally divides into two regimes: $\rho \gg \xi_{\|}$ and $\rho \ll \xi_{\|}$. For $\rho \gg \xi_{\|}$, (51)-(53) imply

$$C_{00}(\rho) \propto \rho^{1-d/2}\, e^{-\rho/\xi_{\|}} \tag{65}$$

for all values of d. For $\rho \ll \xi_{\|}$,

$$C_{00}(\rho) \propto \begin{cases} \bar{\rho}^{3-d} & d>3 \\ -\ln(\bar{\rho}/\xi_{\|}) & d=3 \\ \xi_{\|}^{3-d} & d<3 \end{cases} \tag{66}$$

with $\bar{\rho}=\sqrt{\rho^2+a^2}$. Thus, $C_{00}(0)$ which is related to the interfacial width $\langle \ell^2 \rangle_c$ according to (58) diverges for $d \leqq 3$. If one defines a perpendicular correlation length

$$\xi_{\perp} = \sqrt{\langle \ell^2 \rangle_c} \propto |t|^{-\nu_{\perp}} \tag{67}$$

one obtains from $\langle \ell^2 \rangle_c \simeq C_{00}(0)$ and (66) the scaling relation

$$\nu_{\perp} = \tfrac{1}{2}(3-d)\, \nu_{\|} \tag{68}$$

for $d \leqq 3$ as reported in [18] (note that the formula on p. 170 of [18] contains a misprint: $d-3$ should be replaced by $3-d$). The scaling relation (68) is also fulfilled in SOS-models for the interface fluctuations. Thus, continuous mean-field theory predicts that the interface becomes rough as it delocalizes for $d \leqq 3$. This should be correct if the temperature is above the roughening temperature T_R for the interface between the two coexisting phases [2]. For $T<T_R$, one has to include the effects due to the underlying lattice which should confine the interface fluctuations.

Finally, one should note that the singular part $\bar{C}(\rho, z z')$ as given by (49) depends on the scaling fields not only through $\xi_{\|}$ but also through the factors $g_0(z)\, g_0(z')$. This dependence may be obtained from the asymptotic equality

$$\chi(z z') \to g_0(z)\, g_0(z')/E_0. \tag{69}$$

Consider for instance, the scaling field t. The t-dependence of $\chi(z z')$ and of E_0 is given in Appendix B and by (27), (42), respectively. If this is used in (69), one obtains

$$g_0(z) \propto |t|^{\psi_1} \tag{70}$$

with the surface exponent

$$\psi_1 = \begin{cases} 1 & (P) \\ \beta_1 & (C), (T), (Q) \end{cases} \tag{70.a}$$

for fixed $z \ll \hat{\ell}$ at the critical and multicritical transitions, and for fixed $z \ll z_0$ at the protocritical transition.

I thank D.M. Kroll, H. Spohn, H. Wagner, and R.K.P. Zia for helpful comments. A critical reading of the manuscript by H. Wagner is also gratefully acknowledged.

Appendix A

Here, the bulk and surface order parameters for model (1)-(3) as obtained from Landau theory are discussed. The bulk order parameter follows from $\partial f/\partial\phi|_{M_b}=0$. It is

$$M_b = \begin{cases} M_d \equiv 0 & t>0 \\ M_0 > 0 & t<0 \end{cases} \tag{A.1}$$

with

$$M_0 = \left\{\frac{2+p+\sqrt{p^2-4(1+p)\,t}}{2(1+p)}\right\}^{1/p}. \qquad (A.2)$$

The subscript d and 0 means "disordered" and "ordered" respectively. For $t \to 0^-$, one has

$$M_0 \to 1 - t/p^2 \qquad (A.2a)$$

(note that the amplitude A_0^* has been extracted, compare (1)). In the vicinity of $t=0$, $f(\phi)$ has local minima both at $\phi = M_d$ and at $\phi = M_0$. The local maximum in between occurs at $\phi = \hat{M}$ with

$$\hat{M} = \left\{\frac{2+p-\sqrt{p^2-4(1+p)\,t}}{2(1+p)}\right\}^{1/p}. \qquad (A.3)$$

The local surface order parameter $M_1 \equiv M(z=0)$ derives from

$$f_1'(M_1) = s\sqrt{2f(M_1) - 2f(M_b)} \qquad (A.4)$$

with $s = \pm 1$ for $M_1 \lessgtr M_b$.
A prime indicates a derivative with respect to ϕ. At the critical and multicritical transitions, the quantity

$$\delta M_1 := \begin{cases} M_1 & \text{SID} \\ 1-M & \text{SIO} \end{cases} \qquad (A.5)$$

goes continuously to zero.
First, consider model (1)–(3) with $b_1 = c_1 = 0$. In this case, one finds critical and tricritical transitions as shown in Fig. 1. At $(C^\pm)$, two scaling fields are relevant, namely $|t|$ and

$$\delta h_1 := \begin{cases} h_1 & (C^-) \\ a_1 - h_1 & (C^+). \end{cases} \qquad (A.6)$$

In terms of these fields, δM_1 has the scaling form

$$\delta M_1 = |t|^{\beta_1}\,\Omega_c(|t|^{-\Delta_1}\,\delta h_1). \qquad (A.7)$$

In Landau theory, $\beta_1 = \Delta_1 = 1/2$ both at (C^-) and at (C^+). At the tricritical transitions $(T^\pm)$, one has *three* relevant scaling fields, namely $|t|$, δh_1, and

$$\delta a_1 := \begin{cases} 1/\xi_d^* - a_1 & (T^-) \\ 1/\xi_0^* - a_1 & (T^+). \end{cases} \qquad (A.8)$$

At $(T^\pm)$,

$$\delta M_1 = |t|^{\beta_1}\,\Omega_t(|t|^{-\Delta_1}\,\delta h_1, |t|^{-\phi_a}\,\delta a_1). \qquad (A.9)$$

Landau theory yields $\beta_1 = 1/(p+2)$, $\Delta_1 = (p+1)/(p+2)$, and $\phi_a = p/(p+2)$ at (T^-), and $\beta_1 = 1/3$, $\Delta_1 = 2/3$, and $\phi_a = 1/3$ at (T^-).
For $b_1, c_1 \neq 0$, a higher-order multicritical SID transition (Q^-) is found. (Q^-) occurs for $(h_1, a_1, b_1) = (0, 1/\xi_d^*, 1/\xi_d^*)$ and positive c_1 (for odd p, $c_1 > 0$ is sufficient. For even p, c_1 has to be larger than some positive constant in order to avoid solutions of (A.4) with $M_1 < 0$). (Q^-) is the endpoint of a line of (T^-) transitions (see Fig. 1 in Ref. 18 where (T^-) and (Q^-) have been denoted by $\bar{s}$ and $\bar{s}\,\bar{p}$ respectively).
At (Q^-), one has four relevant scaling fields: $|t|$, $\delta h_1 = h_1$, δa_1 and

$$\delta b_1 := b_1 - 1/\xi_d^*. \qquad (A.10)$$

As a consequence, $\delta M_1 = M_1$ has the scaling form

$$M_1 = |t|^{\beta_1}\,\Omega_q(|t|^{-\Delta_1} h_1, |t|^{-\phi_a}\,\delta a_1, |t|^{-\phi_a}\,\delta b_1) \qquad (A.11)$$

with $\beta_1 = 1/(2p+2)$, $\Delta_1 = (2p+1)/(2p+2)$, $\phi_a = 2p/(2p+2)$, and $\phi_b = p/(2p+2)$ in Landau theory.

Appendix B

In this appendix, the asymptotic behaviour of $\chi(z\,z')$ as given by (46) is studied for different values of z, z'. Only the dependence on the scaling field t is explicitly discussed. The dependence on the other scaling fields may be easily obtained by the same methods.
First, consider critical and multicritical SID. In order to determine the asymptotics of χ, we need the asymptotics of $\dot{M}(z)$, $\bar{B}$, and $v(z)$. The last function is given by

$$v(z) = \int_0^z dx\,[\dot{M}(x)]^{-2} = \int_{M_1}^{M(z)} dm\,[2f(m) - 2f(M_b)]^{-3/2} \qquad (B.1)$$

at SID with

$$-2f(M_b) \to (\xi_d^*)^{-2}\,|t| \qquad (B.2)$$

for $t \to 0^-$. For fixed $z \ll \hat{\ell}$, one has

$$M(z) \to M_1 \cosh(z/\xi_d^*) + (-h_1 + a_1 M_1)\,\xi_d^* \sinh(z/\xi_d^*). \qquad (B.3)$$

As discussed in Appendix A,

$$M_1 \propto |t|^{\beta_1} \qquad (B.4)$$

with $\beta_1 = 1/2$, $1/(p+2)$, and $1/(2p+2)$ at (C^-), (T^-), and (Q^-) respectively. If (B.2)–(B.4) is used in (B.1), one finds

$$v(z) \propto |t|^{-1} \qquad (B.4)$$

at (C^-) and $v(z) \sim o(1/|t|)$ at (T^-), (Q^-) for fixed $z \ll \hat{\ell}$. For $z = \hat{\ell}$ and $M(\hat{\ell}) = \hat{M}$ from (A.3), one obtains

$$v(\hat{\ell}) \to (\xi_d^*)^2 (\xi_d^* - 1/a_1)\,|t|^{-1} \tag{B.6}$$

at (C^-) and $v(\hat{\ell}) \sim o(1/|t|)$ at (T^-), (Q^-). The asymptotic behaviour of $\bar{B}$ which has been defined in (47.b) is given by

$$(\xi_d^*)^2\,\bar{B}/a_1 \to \begin{cases} |t| & (C^-) \\ \frac{p+2}{2}|t| & (T^-) \\ (p+1)\,|t| & (Q^-) \end{cases} \tag{B.7}$$

with $a_1 = 1/\xi_d^*$ at (T^-) and (Q^-). Finally, one needs the asymptotics of $\dot{M}(z)$ which is

$$\dot{M}(\hat{\ell}) = [2f(\hat{M}) - 2f(M_b)]^{1/2} \to [2f^*(\hat{M})]^{1/2} \tag{B.8}$$

and

$$\dot{M}(z) \to (\xi_d^*)^{-1} M_1 \sinh(z/\xi_d^*) + (-h_1 + a_1 M_1)\cosh(z/\xi_d^*) \tag{B.9}$$

for fixed $z \ll \hat{\ell}$ from (B.3). As a result, one obtains

$$\chi(\hat{\ell}\,\hat{\ell}) \to \dot{M}^2(\hat{\ell})\,(\xi_d^*)^3 \times \begin{cases} |t|^{-1} & (C^-) \\ \frac{2}{p+2}|t|^{-1} & (T^-) \\ \frac{1}{p+1}|t|^{-1} & (Q^-) \end{cases} \tag{B.10}$$

from (B.6)–(B.8), and

$$\chi(z\,\hat{\ell}) \propto M_1\,|t|^{-1} \tag{B.11.a}$$

$$\chi(z\,z) \propto M_1^2\,|t|^{-1} \tag{B.11.b}$$

for fixed $z \ll \hat{\ell}$ from (B.5), (B.7), and (B.9).
Next, consider protocritical SIO. In this case, $v(z)$ contains a singular part for $z \geqq z_0$ since $[\dot{M}(z_0)]^{-2} \propto t^{-1}$ from (32.a). After some tedious but straightforward calculations, one obtains at (P^+)

$$v(\hat{\ell}) \to \frac{2}{p}(\xi_d^*)^3\,t^{-1} \tag{B.12.a}$$

$$v(z_0) \to \frac{1}{p}(\xi_d^*)^3\,t^{-1} \tag{B.12.b}$$

and $v(z) \sim O(1)$ for fixed $z \ll z_0$. Since $\bar{B}$ is regular at (P^+),

$$\chi(\hat{\ell}\,\hat{\ell}) \to \dot{M}^2(\hat{\ell})\frac{2}{p}(\xi_d^*)^3\,t^{-1} \tag{B.13.a}$$

$$\chi(z_0\,\hat{\ell}) \to \dot{M}(z_0)\,\dot{M}(\hat{\ell})\,v(z_0) \propto t^{-1/2} \tag{B.13.b}$$

$$\chi(z_0\,z_0) \to \dot{M}^2(z_0)\,v(z_0) = O(1) \tag{B.13.c}$$

$$\chi(z\,z_0) \propto \dot{M}(z_0) \propto t^{1/2} \tag{B.13.d}$$

and $\chi(z\,z) \sim O(1)$ for fixed $z \ll z_0$.

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R. Lipowsky
Sektion Physik
Universität München
Theoretische Physik
Theresienstrasse 37
D-8000 München 2
Federal Republic of Germany