

# Scaling Properties of Interfaces and Membranes

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## 1 Outline and Summary

This review deals with three topics: (i) *Fluctuations* of interfaces and membranes (Sec. 2); (ii) *Interactions* of interfaces and membranes (Sec. 3); and (iii) *Dynamics* of interfaces and growth of wetting layers (Sec. 4). All three topics are intimately related: fluctuations renormalize the interactions, and the interactions act as a driving force for the dynamics.

I will focus on interfaces and membranes which are *rough* but not (yet) crumpled. The scaling properties of such surfaces are governed by the roughness exponent,  $\zeta$  (Sec. 2.2). For *interfaces*, this exponent depends on the nature of the two phases separated by the interface which can be fluids, periodic crystals, quasicrystals, or random systems (Sec. 2.3). For *membranes*,  $\zeta$  depends on the internal membrane structure which may be fluid, crystalline, or hexatic (Sec. 2.4). In all cases, the fluctuations give a singular contribution to the surface free energy, see Sec. 2.5.

The effective *interaction* of surfaces represents a *unifying concept* by which one can understand many different phenomena, see Sec. 3.1 and 3.2. This point of view is not entirely new: there exists a huge literature on interactions between *planar* surfaces (Sec. 3.3). However, it has been realized only recently that these interactions are often renormalized by *fluctuations*. Several scaling regimes for the renormalized interaction must be distinguished, see Sec. 3.4. These different regimes can be studied in a systematic way, starting from an effective Hamiltonian for the interacting surfaces (Sec. 3.5). So far, the most useful method has been a *functional renormalization group* (RG), see Sec. 3.6. As a result, one finds RG fixed points for the critical behavior associated with the unbinding of surfaces (Sec. 3.7 and 3.8).

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Unbinding occurs for repulsive interactions which drive the surfaces apart. The dynamics of this process is discussed in Sec. 4. Several growth modes for interfaces are considered: adhesive growth (Sec. 4.1 and 4.2), diffusion-limited growth (Sec. 4.3), and activated growth resulting from quenched impurities (Sec. 4.4). These growth modes are relevant for the thickening of wetting layers or thin films.

## 2 Fluctuations of Interfaces and Membranes

### 2.1 Introduction

An interface or domain wall represents the contact region between two bulk phases of matter. This contact region has an intrinsic width which is usually microscopic and set by the bulk correlation lengths of the two phases. At finite temperatures,  $T > 0$ , the interface undergoes thermally-excited fluctuations which lead to a certain interfacial roughness. The interface may also be roughened by quenched impurities which provide an effective random potential. These fluctuations change the area of the interface and, thus, are controlled by *tension*.

Membranes are sheets of amphiphilic molecules which form spontaneously when these molecules are dissolved in water or in mixtures of water and oil. Such membranes play an essential role in biophysics since they provide the spatial organization of cells and organelles. For  $T > 0$ , membranes are also deformed by thermal fluctuations but typical undulations are *bending* modes controlled by *curvature*.

In this section, I will discuss the properties of a *single* interface or membrane. First, the roughness of surface fluctuations is defined in a precise way and the basic *roughness exponent*,  $\zeta$ , is introduced (Sec. 2.2). The possible values of  $\zeta$  are then summarized in Sec. 2.3 and Sec. 2.4 for interfaces and membranes, respectively. Finally, the free energy arising from the surface fluctuations is discussed in Sec. 2.5. This free energy has a scaling form which is characterized by the *decay exponent*,  $\tau$ .

### 2.2 Scale invariance of surfaces and the roughness exponent $\zeta$

Consider a 2-dim surface segment with linear size  $\sim L$  which is embedded in 3-dim space. At  $T > 0$ , the surface segment will make bumps in order to increase its configurational entropy. The bumpy surface can be characterized by two length scales,  $L_{\parallel}$  and  $L_{\perp}$ . The scale  $L_{\parallel}$  gives the area  $\sim L_{\parallel}^2$  of a typical surface configuration when *projected* onto the planar reference state; the scale  $L_{\perp}$  measures the average distance of the surface from this reference state. In general, I will consider (d-1)-dim surfaces with projected area  $\sim L_{\parallel}^{d-1}$  which fluctuate in d-dim space. Two examples in  $d = 1 + 1$  are: (a) domain boundaries within an

adsorbed monolayer, and (b) steps or ledges on a crystal surface.

The two length scales  $L_{\parallel}$  and  $L_{\perp}$  are not independent but satisfy the scaling relation  $L_{\perp} \sim L_{\parallel}^{\zeta}$  where  $\zeta \geq 0$  is the *roughness exponent*.<sup>1,2</sup> Several cases must be distinguished: (i) A *smooth* surface with  $\zeta = 0$ ; in this case, the size of  $L_{\perp}$  is set by a microscopic length scale (or cutoff),  $a$ ; (ii) A *marginally rough* surface with  $\zeta = 0(\sqrt{\log})$  which corresponds to

$$L_{\perp} \sim [\ln(L_{\parallel}/a)]^{1/2} \quad \text{for} \quad L_{\parallel} \gg a. \quad (2.1)$$

(iii) A *rough* surface with  $0 < \zeta < 1$  and

$$L_{\perp} \sim L_{\parallel}^{\zeta} \quad \text{for} \quad L_{\parallel} \gg a. \quad (2.2)$$

(iv) A rough surface with  $\zeta \geq 1$ ; in this case, the scaling relation between  $L_{\perp}$  and  $L_{\parallel}$  holds only on *intermediate* length scales:

$$L_{\perp} \sim L_{\parallel}^{\zeta} \quad \text{for} \quad \xi_p \gg L_{\parallel} \gg a \quad (2.3)$$

where  $\xi_p$  is the *persistence length* of the surface. This length which was introduced in the context of membranes<sup>3</sup> gives the correlation length of the surface normals. For  $L_{\parallel} \gg \xi_p$ , the normals decorrelate and the surface is *crumpled*.<sup>3</sup> For  $\zeta < 1$ , one always has  $\xi_p = \infty$ .

Thermal fluctuations provide an *entropic* mechanism for the roughening of surfaces. Alternatively, the surface can be roughened by an external potential. In this case, it tries to minimize its *energy* by adapting its shape to the minima of the external potential. The latter mechanism underlies, e.g., the roughening by quenched random impurities.

So far, a finite surface segment has been considered. Now, take the thermodynamic limit,  $L_{\parallel} \rightarrow \infty$ . Then, the roughness exponent  $\zeta$  can be extracted from the difference correlation function,  $\Delta C(x) \equiv \langle [\ell(\vec{x}) - \ell(\vec{0})]^2 \rangle$ , for the variable  $\ell(\vec{x})$  which measures the distance of the fluctuating surface from its planar reference state with coordinate  $\vec{x} = (x_1, \dots, x_{d-1})$ . Indeed, the scaling relation (2.2) implies<sup>1</sup>

$$\Delta C(x) \sim x^{2\zeta} \quad \text{or} \quad \Delta C(x) \approx b^{2\zeta} \Delta C(x/b) \quad \text{for} \quad x \gg a. \quad (2.4)$$

Thus, the bumpy surface is invariant under the rescaling transformation  $x \rightarrow x/b$  and  $\ell \rightarrow \ell/b^{\zeta}$ .

## 2.3 Roughness of interfaces

**1. Fluid phases and periodic crystals** — An interface separating two fluid phases is characterized by the roughness exponent  $\zeta = \zeta_o$  with

$$\begin{aligned} \zeta_o &= 0(\sqrt{\log}) & d = 3 \\ &= (3-d)/2 & 1 < d < 3. \end{aligned} \quad (2.5)$$

The same value applies to an interface which separates a periodic crystal from another phase *provided*  $T$  exceeds the roughening temperature,  $T_R$ .

**2. Ideal quasicrystals** — In this case, the interface feels a quasiperiodic potential which stiffens the interface and *decreases* the value of  $\zeta$ . A Fibonacci potential in  $d = 2$ , which approximates the ideal Penrose tiling, leads to the *non-universal* value<sup>4,5</sup>

$$\zeta < \zeta_o = 1/2 \quad \text{and} \quad \zeta \approx 2 \ln[(1 + \sqrt{5})/2] T/J \quad \text{for small } T \quad . \quad (2.6)$$

For Harper's potential, the interface undergoes a roughening transition in  $d = 2$ .<sup>4</sup>

In general, the large scale properties of a rough interface with roughness exponent  $0 < \zeta \leq \zeta_o = (3 - d)/2$  can be described by the *scale-dependent stiffness*

$$\tilde{\Sigma}_{eff}(L_{\parallel}) \sim L_{\parallel}^{\eta_{\sigma}} \quad \text{with} \quad \eta_{\sigma} = 2(\zeta_o - \zeta) \geq 0 \quad . \quad (2.7)$$

In fluids and in periodic crystals with  $T > T_R$ , the exponent  $\eta_{\sigma} = 0$  and  $\tilde{\Sigma}_{eff}(L_{\parallel} = \infty) \equiv \tilde{\Sigma}$  is finite. A finite  $\tilde{\Sigma}$  also applies to random systems with  $\zeta < 1$ .

**3. Random quasicrystals** – Random tilings can be obtained from the ideal tiling by a random rearrangement of the tiles thereby abandoning the matching rules.<sup>6,7</sup> Then, the interfacial roughness is *enhanced* compared to a periodic system, and<sup>8</sup>

$$\zeta = 2/3 \quad \text{in} \quad d = 1 + 1 \quad \text{for} \quad T > 0 \quad . \quad (2.8)$$

For  $T = 0$ , the interface is marginally rough and smooth in  $d = 1+1$  and  $d = 2+1$ , respectively.<sup>8</sup>

**4. Quenched random impurities** – Two types of impurities must be distinguished; (i) Random fields which couple directly to the order parameter density, and (ii) Random bonds which couple to the energy density. For *random fields*, a scaling argument<sup>9</sup> leads to<sup>10–12</sup>

$$\begin{aligned} \zeta &= 0 & \text{for} & \quad d > 5 \\ &= (5 - d)/3 & \text{for} & \quad 2 < d < 5 \quad . \end{aligned} \quad (2.9)$$

For *random bonds*, the corresponding scaling argument fails but

$$\zeta = 2/3 \quad \text{in} \quad d = 1 + 1 \quad . \quad (2.10)$$

is known exactly both for  $T = 0$  and for  $T > 0$ .<sup>13–16</sup>

## 2.4 Roughness of membranes

**1. Fluid membranes** – For fluid membranes such as lipid bilayers in the  $L_{\alpha}$  phase,<sup>17</sup> the elastic bending energy is governed by the mean curvature.<sup>18</sup> This leads to  $\zeta = \zeta_o$  with<sup>19</sup>

$$\begin{aligned} \zeta_o &= 0(\sqrt{\log}) & \text{for} & \quad d = 5 \\ &= (5 - d)/2 & \text{for} & \quad 1 < d < 5 \quad . \end{aligned} \quad (2.11)$$

Thus, in  $d = 3$ ,  $\zeta = 1$ ,<sup>20</sup> and the persistence length,  $\xi_p$ , is finite<sup>3</sup>.

**2. Crystalline and polymerized membranes** – Lipid bilayers usually exhibit  $L_\beta$  phases at low T in which the lipid molecules form a lattice or network.<sup>17</sup> A similar situation occurs for membranes composed of polymerizable lipids which provide models for biological systems.<sup>21</sup> Then, the bending modes are coupled to the internal degrees of freedom.<sup>22</sup> At low T, these membranes exhibit a *scale-dependent bending rigidity*:<sup>22,23</sup>

$$\kappa_{eff}(L_\parallel) \sim L_\parallel^{\eta_\kappa} \quad \text{with} \quad 2/3 \leq \eta_\kappa \leq 1 \quad \text{in} \quad d = 2 + 1 . \quad (2.12)$$

In general, a scale-dependent rigidity implies

$$\zeta = \zeta_o - \eta_\kappa/2 \quad (2.13)$$

with  $\zeta_o$  as given by (2.11).

**3. Hexatic membranes** – The crystalline membrane could melt into a hexatic one if the free energy of a single dislocation is finite as a result of buckling.<sup>22</sup> At low T, hexatic membranes acquire a scale-independent rigidity<sup>24</sup> which implies

$$\eta_\kappa = 0 \quad \text{and} \quad \zeta = 1 \quad \text{in} \quad d = 2 + 1 . \quad (2.14)$$

## 2.5 Scaling of surface free energy

The surface fluctuations give a singular contribution,  $\Delta f$ , to the surface free energy. Consider again a surface segment with projected area  $\sim L_\parallel^{d-1}$ , and assume that the surface is ‘clamped’ to a reference plane along one edge. The opposite edge is free and will make transverse excursions  $\sim L_\perp$  from the reference plane. These largest humps have wavenumber  $\sim 1/L_\parallel$  and should contain a thermal free energy  $\sim T$  as suggested by the equipartition theorem. This gives an entropic contribution,  $\Delta s$ , to the free energy per unit (projected) area which scales as

$$-T\Delta s \sim T/L_\parallel^{d-1} . \quad (2.15)$$

On the other hand, the elastic free energy per unit area,  $\Delta e$ , behaves as

$$\Delta e \sim \tilde{\Sigma}_{eff} (L_\perp/L_\parallel)^2 \sim L_\parallel^{\eta_\sigma} (L_\perp/L_\parallel)^2 \quad (2.16)$$

for interfaces, and as

$$\Delta e \sim \kappa_{eff} (L_\perp/L_\parallel^2)^2 \sim L_\parallel^{\eta_\kappa} (L_\perp/L_\parallel^2)^2 \quad (2.17)$$

for membranes. Therefore,  $\Delta e$  has the general scaling form

$$\Delta e \sim (L_\perp/L_\parallel^n)^2 \quad (2.18)$$

with

$$\begin{aligned} n &= 1 - \eta_\sigma/2 && \text{for interfaces} \\ &= 2 - \eta_\kappa/2 && \text{for membranes.} \end{aligned} \quad (2.19)$$

Now, the free energy per unit area,  $\Delta f$ , is taken to be  $\Delta f = \Delta e - T\Delta s$ .<sup>25,8</sup> For a marginally rough surface with  $\zeta = 0(\sqrt{\log})$ , the elastic free energy dominates and

$$\Delta f \sim \ln(L_{\parallel}/a)/L_{\parallel}^{2n} \sim L_{\perp}^2 \exp\{-2nL_{\perp}^2/a_{\perp}^2\} \quad (2.20)$$

where  $a_{\perp}$  is a microscopic length scale which must be distinguished from  $a$ <sup>26</sup>. For a rough surface with  $\zeta > 0$  (and  $a \ll L_{\parallel} \ll \xi_p$ ),  $\Delta f$  exhibits the power law behavior

$$\Delta f \sim 1/L_{\parallel}^{\zeta\tau} \sim 1/L_{\perp}^{\tau} \quad (2.21)$$

The *decay exponent*,  $\tau$ , depends on  $\zeta$ ,  $d$ , and  $n$ . Two cases must be distinguished:

- (i) For thermally-excited surface fluctuations, one usually has  $\zeta = (2n + 1 - d)/2$ . Then, the entropic contribution and the elastic free energy have the same order of magnitude, and

$$\tau = 2(n/\zeta - 1) = (d - 1)/\zeta . \quad (2.22)$$

- (ii) For *interface* fluctuations induced by a random potential, the roughness exponent satisfies  $\zeta > (2n + 1 - d)/2$  with  $n = 1$ , and the elastic free energy scales as the energy gain resulting from the potential. In this case, the elastic free energy dominates and<sup>1</sup>

$$\tau = 2(1/\zeta - 1) < (d - 1)/\zeta . \quad (2.23)$$

## 3 Interactions of Interfaces and Membranes

### 3.1 Introduction

In the previous section, a single interface or membrane has been considered. In real systems, one usually has two or several surfaces rather than a single one. In some cases, the surfaces are *oriented* in such a way that they are, on average, *parallel*. Condensed matter physics provides several examples for such a behavior where the surfaces are 1- or 2-dimensional:

- (a) Wetting, surface melting, and surface-induced disorder<sup>27</sup> — In this case, the contact region of two bulk phases,  $\alpha$  and  $\gamma$ , contains a thin film or layer of a third phase,  $\beta$ . This film is bounded by two interfaces which have the same average orientation. Surface melting<sup>28–31</sup>, edge melting<sup>32</sup>, and surface-induced disorder<sup>33–35</sup> have recently been studied by a variety of experimental methods.
- (b) Equilibrium shape of crystals — This shape is determined, to a large extent, by the behavior of the steps or ledges which separate flat terraces of the crystal surface.<sup>36</sup> The average orientation of these steps is determined by the index of the crystal surface.

- (c) Commensurate–Incommensurate transitions — Near such a transition, the system consists of commensurate domains which are separated by domain walls. In many cases, these walls (or lines) are parallel and form a striped phase, see Ref. 2 and references therein.
- (d) Adhesion of vesicles or biological cells — A vesicle composed of an amphiphilic membrane can adhere to another surface (which may be a second membrane). Within the region of contact, the membrane and the surface are roughly parallel.<sup>17</sup>
- (e) Lamellar phases of membranes — Such phases occur (i) in binary systems of water and lipids, and (ii) in oil–water mixtures containing amphiphilic molecules. They are composed of a stack of membranes which have the same average orientation. For recent experimental studies, see Refs. 37 and 38.

In all of these examples, the surfaces experience a mutual interaction. In this section, I discuss the nature and the form of this interaction. First, the *total* interaction,  $V_{TI}(\ell)$ , of two surfaces with separation  $\ell$  is defined in Sec. 3.2. In the *absence* of surface fluctuations, this interaction reflects the intermolecular forces and is called the *direct* interaction,  $V_{DI}(\ell)$ , see Sec. 3.3.

Surface fluctuations lead to a nontrivial *renormalization* of  $V_{DI}(\ell)$ . Several scaling regimes (or universality classes) must be distinguished (Sec. 3.4). In general, one must include fluctuations on *all* length scales. This can be done starting from a systematic theory, see Sec. 3.5. So far, the most powerful method has been a nonperturbative functional renormalization group (RG). The results of this RG will be described in Sec. 3.6. Finally, the critical behavior associated with the unbinding of surfaces will be discussed in Sec. 3.7 and 3.8.

## 3.2 External pressure and total interaction of surfaces

Consider two roughly parallel surfaces (or surface segments) and assume that one can change the separation,  $\ell$ , of these surfaces by an external pressure,  $H$ . I will use the sign convention that *increasing* pressure leads to *decreasing* surface separation.

In the context of wetting, a thin film of a metastable phase,  $\beta$ , intrudes between two coexisting bulk phases,  $\alpha$  and  $\gamma$ . Then, the free energy per unit volume,  $\hat{f}_\beta$ , of the  $\beta$  phase exceeds the free energy per unit volume,  $f_\alpha = f_\gamma = f_{\alpha\gamma}$ , of the coexisting phases  $\alpha$  and  $\gamma$ , and the external pressure is given by  $H = \hat{f}_\beta - f_{\alpha\gamma}$ .<sup>27</sup> Thus, for wetting,  $H$  can be experimentally controlled via the temperature,  $T$ , and via the chemical potentials of the molecular species. For membranes,  $H$  can be controlled by an osmotic or hydrostatic pressure, or by changing a relative humidity.<sup>39</sup>

The external pressure  $H$  is balanced by a disjoining pressure,  $-\partial V_{TI}/\partial\ell$ , arising from the total interaction,  $V_{TI}(\ell)$ , of the surfaces.<sup>40</sup> Thus, the mean separa-

tion,  $\bar{\ell}$ , is determined by

$$H = -\partial V_{TI}/\partial \ell \quad \text{for} \quad \ell = \bar{\ell}. \quad (3.1)$$

In an experiment (or in a computer simulation), one can measure the mean surface separation,  $\bar{\ell}$ , as a function of  $H$ . Then, the total interaction,  $V_{TI}(\ell)$ , can be obtained via a Legendre transformation from  $H$  to  $\ell$ .<sup>27</sup> First, one inverts the relation  $\ell = \bar{\ell}(H)$  in order to obtain  $H = \bar{H}(\ell)$ . When this is inserted into (3.1), one has  $-\partial V_{TI}/\partial \ell = \bar{H}(\ell)$  which determines  $V_{TI}(\ell)$  up to a constant. This shows that the total interaction,  $V_{TI}(\ell)$ , is a well-defined quantity which contains the same information as  $\ell = \bar{\ell}(H)$ . Now, from a *theoretical* point of view, one would like to *predict* the behavior of  $\ell = \bar{\ell}(H)$  and, thus, of  $V_{TI}(\ell)$ . This can be done in two steps: (i) First, one ignores surface fluctuations and studies the *direct* interaction,  $V_{DI}(\ell)$ , between planar surfaces<sup>41</sup>; (ii) Secondly, one takes surface fluctuations into account which renormalize  $V_{DI}(\ell)$ . *This renormalized interaction represents the theoretical prediction for  $V_{TI}(\ell)$ .*

### 3.3 Direct interactions of planar surfaces

The direct interactions,  $V_{DI}(\ell)$ , depend on the microscopic forces acting between the molecules. Two contributions have been known for a long time:<sup>42</sup> (i) *van der Waals interactions* — In  $d = 3$ , this interaction decays as  $\sim 1/\ell^2$  and  $1/\ell^3$  for nonretarded and retarded forces, respectively. (ii) *Electrostatic interactions* — Interfaces or membranes often contain electric charges. If the surface separation  $\ell$  is large compared to the Debye screening length,  $\lambda_E$ , the associated direct interaction decays as  $\sim \exp(-\ell/\lambda_E)$ . For  $a \ll \ell \ll \lambda_E$ ,  $V_{DI}$  exhibits a power law behavior.

More recently, the direct interaction between surfaces has been studied within Landau or van der Waals theories. As a result, one finds several cases depending on the nature of the phase,  $\beta$ , between the surfaces: (iii) *Exponential interactions* — If the intervening  $\beta$  phase is composed of small molecules and has a microscopic correlation length,  $\xi_\beta$ ,  $V_{DI} \sim \exp(-\ell/\xi_\beta)$ .<sup>43–45</sup> (iv) *Interactions induced by critical fluctuations* — If the intervening  $\beta$  phase is critical or near-critical, scaling theory implies  $V_{DI} \sim 1/\ell^{d-1}$ .<sup>46</sup> (v) *Polymer-induced interactions* — When the surfaces are covered by adsorbed polymers, one has additional direct interactions which decay like a power of  $\ell$ .<sup>47</sup>

### 3.4 Different scaling regimes for the unbinding of surfaces

Now, consider a *fluctuating* surface which is bound, via attractive direct interactions, to another surface. In such a situation, the roughness of the fluctuating surface is restricted by the presence of the second surface. The size,  $\xi_\perp$ , of this roughness is set by the transverse extension of the largest bumps. For smaller bumps with roughness  $L_\perp \ll \xi_\perp$ , the fluctuations are essentially unrestricted, and their lateral extension,  $L_\parallel$ , scales as in (2.1)–(2.3). Therefore, the largest

bumps can be characterized by a longitudinal correlation length,  $\xi_{\parallel}$ , which satisfies  $\xi_{\perp} \sim [\ln(\xi_{\parallel}/a)]^{1/2}$  for  $\zeta = 0(\sqrt{\log})$  and  $\xi_{\perp} \sim \xi_{\parallel}^{\zeta}$  for  $\zeta > 0$ .

The two length scales,  $\xi_{\parallel}$  and  $\xi_{\perp}$ , play essentially the same role for the bound surfaces as the scales  $L_{\parallel}$  and  $L_{\perp}$  for a finite surface segment, compare Sec. 2.2. Therefore, the excess free energy,  $V_{FL}(\xi_{\perp})$ , of these largest bumps can be estimated as in Sec. 2.5:<sup>25</sup>

$$V_{FL}(\xi_{\perp}) \sim 1/\xi_{\perp}^{\tau} \quad \text{with} \quad \tau = 2(n/\zeta - 1) \quad (3.2)$$

for a rough surface with  $\zeta > 0$ , and

$$V_{FL}(\xi_{\perp}) \sim \xi_{\perp}^2 \exp[-2n\xi_{\perp}^2/a_{\perp}^2] \quad (3.3)$$

for a marginally rough surface, compare (2.21) and (2.20).

This excess free energy can be interpreted as a *fluctuation-induced repulsion* acting between the surfaces. It can be used in order to identify different scaling regimes for the *unbinding* of surfaces. In the process of unbinding, the mean separation,  $\bar{\ell}$ , becomes large compared to microscopic length scales, and the surfaces probe the tail of their direct interaction,  $V_{DI}$ . Therefore, each scaling regime contains a certain *class* of direct interactions.

**1. Rough surfaces with  $\zeta > 0$**  — In this case, one must distinguish four scaling regimes: a *mean-field* (MF) regime and three different *fluctuation* (FL) regimes. The two length scales,  $\bar{\ell}$  and  $\xi_{\perp}$ , behave as<sup>48</sup>

$$\bar{\ell} \gg \xi_{\perp} \quad \text{in MF regime} \quad \text{and} \quad \bar{\ell} \sim \xi_{\perp} \quad \text{in FL regimes} . \quad (3.4)$$

(i) The MF regime is characterized by<sup>1,25</sup>

$$V_{DR}(\ell) \gg V_{FL}(\ell) \sim 1/\ell^{\tau} \quad \text{for large } \ell \quad (3.5)$$

where  $V_{DR}$  represents the *repulsive* part of  $V_{DI}$ . (ii) The *weak-fluctuation* (WFL) regime is defined by<sup>1,25</sup>

$$V_{DR}(\ell) \ll V_{FL}(\ell) \sim 1/\ell^{\tau} \ll V_{DA}(\ell) \quad \text{for large } \ell \quad (3.6)$$

where  $V_{DA}$  represents the *attractive* part of  $V_{DI}$ . In addition, one has two non-trivial fluctuation regimes: (iii) the *intermediate fluctuation* (IFL) regime with<sup>49</sup>

$$|V_{DI}(\ell)| \sim V_{FL}(\ell) \sim 1/\ell^{\tau} \quad \text{for large } \ell \quad , \quad (3.7)$$

which contains, in fact, three different subregimes; and (iv) the *strong-fluctuation* (SFL) regime characterized by<sup>1,25</sup>

$$|V_{DI}(\ell)| \ll V_{FL}(\ell) \sim 1/\ell^{\tau} \quad \text{for large } \ell \quad . \quad (3.8)$$

**2. Marginally rough surfaces with  $\zeta = 0(\sqrt{\log})$**  — In this case, the two length scales,  $\bar{\ell}$  and  $\xi_{\perp}$ , satisfy

$$\bar{\ell} \gg \xi_{\perp}^2 \quad \text{in MF regime} \quad \text{but} \quad \bar{\ell} \sim \xi_{\perp}^2 \quad \text{in FL regimes} . \quad (3.9)$$

The MF regime is now characterized by  $V_{DR}(\ell) \gg V_{FL}(\sqrt{\ell}) \sim \exp(-c\ell/a)$  for large  $\ell$ . Thus, one enters the FL regimes for a short-ranged repulsion  $V_{DR} \sim \exp(-c\ell/a)$ .

For  $V_{DI}(\ell) \approx -W/\ell^r + U\exp(-\ell/a_1)$ , the linear renormalization group (RG) introduced in Ref. 50 leads to two subregimes depending on the size of the microscopic length scale,  $a_1$ .<sup>51</sup> For  $V_{DI}(\ell) \approx -W\exp(-\ell/a_1) + U\exp(-\ell/a_2)$ , this linear RG yields three subregimes<sup>50,52</sup> as has been confirmed, to a certain extent, by MC simulations of interface models<sup>53</sup>. Exponential interactions should apply to 3-dim lattice models with short-ranged interactions. However, MC simulations of such models have not produced, so far, any evidence for the FL regimes.<sup>54,55</sup>

### 3.5 Effective Hamiltonian for interacting surfaces

Consider two surface segments which are, on average, parallel and are described by two coordinates,  $\ell_1(\vec{x})$  and  $\ell_2(\vec{x})$ . For thermally-excited fluctuations their elastic free energy per unit area is given by  $\frac{1}{2}K_1(\nabla^n \ell_1)^2$  and  $\frac{1}{2}K_2(\nabla^n \ell_2)^2$ , respectively, with  $n$  as in (2.19) and  $\zeta = (2n + 1 - d)/2$ . Furthermore, the two surfaces experience a direct interaction,  $V_{DI}(\ell_1 - \ell_2)$ . Then, their separation  $\ell \equiv \ell_1 - \ell_2 \geq 0$  is governed by the effective Hamiltonian

$$\mathcal{H}\{\ell\} = \int d^{d-1}x \left\{ \frac{1}{2}K(\nabla^n \ell)^2 + V(\ell) \right\} \quad (3.10)$$

with  $K = K_1 K_2 / (K_1 + K_2)$  and  $V(\ell) \equiv H\ell + V_{DI}(\ell)$  for  $\ell > 0$ .

The field  $\ell$  is restricted to positive values since the surfaces considered here cannot intersect. Therefore, the model (3.10) should be supplemented with the *hard wall* condition

$$V(\ell) = \infty \quad \text{for} \quad \ell < 0. \quad (3.11)$$

Then, all configurations with negative  $\ell$ -values have a vanishing Boltzmann weight,  $\exp\{-\mathcal{H}/T\}$ . For  $\ell > 0$ , the interaction  $V(\ell)$  has the generic form  $V(\ell) = H\ell + V_{DI}(\ell)$  with  $V_{DI}(\ell) \approx 0$  for large  $\ell$ . For effective pressure  $H > 0$ , the term  $H\ell$  provides an exponential cutoff in the weight for large  $\ell$ . For  $H = 0$ , configurations with arbitrarily large  $\ell$  have a finite weight. Therefore, the surfaces are always *unbound* at  $H = 0$  as long as the number of surface modes is finite ( $L_{\parallel} < \infty$ ). In order to get a bound state of the surfaces, one must *first* perform the thermodynamic limit ( $L_{\parallel} \rightarrow \infty$ ) for  $H > 0$ , and subsequently let  $H$  approach zero.<sup>27</sup>

The model (3.10) can be studied with a variety of theoretical methods.<sup>27</sup> So far, the only approach which has been useful for general  $d$  and  $n$  is a functional renormalization group (RG)<sup>56,19,25</sup> which represents an extension of Wilson's approximate recursion relations<sup>57</sup>.

### 3.6 Functional renormalization

**1. Strong-fluctuation (SFL) regime** — The functional RG acts as a *nonlinear* map on the direct interaction,  $V(\ell) = V_{DI}(\ell)$ , of interfaces or membranes. When

applied to the model given by (3.10), this RG leads to the recursion relation<sup>56,19,25</sup>

$$V^{(N+1)}(\ell) = R[V^{(N)}(\ell)] \quad (3.12)$$

with

$$R[V(\ell)] = -\tilde{v}b^{d-1}\ln\left\{\int_{-\infty}^{\infty}\frac{d\ell'}{\sqrt{2\pi\tilde{a}_{\perp}}}\exp\left[-\frac{1}{2}(\ell'/\tilde{a}_{\perp})^2 - G(\ell, \ell')\right]\right\} \quad (3.13)$$

and

$$G(\ell, \ell') = [V(b^{\zeta}\ell - \ell') + V(b^{\zeta}\ell + \ell')]/2\tilde{v}. \quad (3.14)$$

The parameters  $\tilde{a}_{\perp}$  and  $\tilde{v}$  which depend on the rescaling factor  $b > 1$  represent a length and a free energy scale. The length scale  $\tilde{a}_{\perp}$  is determined by the requirement that the RG transformation is exact to first order in  $V$ . This leads to

$$\tilde{a}_{\perp}^2 = a_{\perp}^2(b^{2\zeta} - 1)/2\zeta \quad \text{with} \quad a_{\perp}^2 = c_d(T/K)a^{2\zeta}, \quad (3.15)$$

$c_d = 2/(4\pi)^{(d-1)/2}\Gamma(\frac{d-1}{2})$ , and  $\zeta = (2n + 1 - d)/2$  as before. The scale  $\tilde{a}_{\perp}$  has a simple interpretation: it is the roughness of the small-scale fluctuations with wavelengths  $a \leq L_{\parallel} \leq ba$ . The free energy scale,  $\tilde{v}$ , is not determined by the linearized RG. Wilson's original decomposition of phase space leads to the choice

$$\tilde{v} = v(1 - b^{1-d})/(d - 1) \quad \text{with} \quad v = c_d T/a^{d-1}. \quad (3.16)$$

For rough surfaces with  $\zeta = (2n + 1 - d)/2 > 0$ , the above recursion relation leads to two fixed points,  $V_o^*(\ell)$  and  $V_c^*(\ell)$ ,<sup>56,19,25</sup> which have a Gaussian tail  $\sim \exp(-\ell^2)$ . These two fixed points govern the behavior within the SFL regime as defined by (3.8). The fixed point  $V_o^*(\ell)$  is purely repulsive and describes the unbound state of the surfaces. All direct interactions,  $V(\ell) = V_{DI}(\ell)$ , which lead to completely separated surfaces are mapped onto this fixed point. On the other hand, the fixed point  $V_c^*(\ell)$  has an attractive well and describes the continuous unbinding transition within the SFL regime. All direct interactions which correspond to such a transition point are mapped onto  $V_c^*$ . Thus, within the SFL regime, the fixed points  $V_o^*$  and  $V_c^*$  have a domain of attraction with codimension zero and one, respectively.

**2. Intermediate (IFL) regime**—Now, consider interactions  $V(\ell) = V_{DI}(\ell)$  which belong to the IFL regime and decay as  $\sim 1/\ell^{\tau}$  for large  $\ell$ . This regime is governed by a *whole line* of nontrivial RG fixed points as found from the functional RG in the infinitesimal rescaling limit  $b \rightarrow 1 + \Delta t$ .<sup>58</sup> In this limit, the RG transformation (3.12) becomes<sup>25</sup>

$$\partial V/\partial t = (d - 1)V + \zeta\ell\partial V/\partial\ell + \frac{1}{2}v \ln[1 + (a_{\perp}^2/v)\partial^2 V/\partial\ell^2] \quad (3.17)$$

with  $a_{\perp}$  and  $v$  as in (3.15) and (3.16).

It is convenient to use the dimensionless variables  $z \equiv \sqrt{2\zeta}\ell/a_{\perp}$  and  $U(z) \equiv 2\zeta V(a_{\perp}z/\sqrt{2\zeta})/v$ . Then, the flow equation becomes<sup>58</sup>

$$\partial U/\partial t = \zeta[\tau U + zU' + \ln(1 + U'')]. \quad (3.18)$$

The fixed points,  $U^*(z)$ , of this RG transformation satisfy

$$\tau U^* + z \partial U^* / \partial z + \ln[1 + \partial^2 U^* / \partial z^2] = 0 . \quad (3.19)$$

Therefore, the rescaled fixed point equation depends only on *one* parameter, namely  $\tau$ .

This fixed point equation has solutions which are singular at  $z = 0$  and behave as<sup>58</sup>

$$U^*(z) \approx \sigma / z^\tau + \frac{\tau + 2}{\tau} \ln(z) \quad \text{with } \sigma > 0 \quad (3.20)$$

for *small*  $z$ . For *large*  $z$ , all solutions to (3.19) decay, and  $U^*$  is then governed by the linear equation,  $\tau U^* + z \partial U^* / \partial z + \partial^2 U^* / \partial z^2 = 0$ . This implies<sup>58</sup>

$$U^*(z) \approx \rho(\sigma) / z^\tau + \bar{\rho}(\sigma) z^{\tau-1} \exp(-z^2/2) \quad (3.21)$$

for large  $z$  where the amplitudes  $\rho$  and  $\bar{\rho}$  are uniquely determined by  $\sigma$ . This line of RG fixed points,  $U^*(z|\sigma)$ , leads to non-universal critical behavior and essential singularities in agreement with exact calculations<sup>49</sup> for  $(d, n) = (2, 1)$ . The corresponding RG flow is unusual and has a parabolic character.

**3. Fluctuation regimes for marginally rough surfaces** — For the marginal case  $\zeta = (2n + 1 - d)/2 = 0$ , the flow equation (3.17) reduces to

$$\partial V / \partial t = 2nV + \frac{1}{2}v \ln[1 + (a_\perp^2/v) \partial^2 V / \partial \ell^2] \quad (3.22)$$

Then, the equation for the fixed points,  $V^*(\ell)$ , can be written as  $\partial^2 V^* / \partial \ell^2 = -\partial \Phi(V^*) / \partial V^*$  where  $\Phi(V^*)$  has a unique minimum at  $V^* = 0$ .<sup>25</sup> This implies that one has *no* (stationary) unbinding fixed points for  $\zeta = 0$ . Instead, a line of *drifting* fixed points,  $V(\ell, t) = V^\dagger(\ell - gt)$ , is found which move with constant velocity,  $g$ , under the RG.<sup>56,25</sup> They arise because the redundant perturbation,  $\partial V^* / \partial \ell$ , associated with a shift of the  $\ell$ -coordinate, becomes marginal for  $\zeta = 0$ .

In the absence of a nontrivial fixed point, the critical behavior must be determined by a matching procedure: one applies the RG up to a matching point at which the renormalized interaction,  $V^{(N)}(\ell)$  or  $V(\ell, t)$ , can be analyzed by mean-field theory. Such a matching procedure can be done analytically if one *linearizes* the RG transformation. If the recursion relation (3.13) is linearized as it stands, one obtains a linear RG for  $V(\ell)$ <sup>52,25</sup> which is completely equivalent to a normal ordering of  $V(\ell)$ <sup>59,48,27</sup>. On the other hand, one may first incorporate the hard wall condition (3.11) into (3.13), and *subsequently* linearize. The latter procedure leads to the modified linear RG introduced in Ref. 50.

### 3.7 Critical behavior at unbinding transitions

The critical behavior associated with the unbinding of surfaces can be characterized by critical exponents. First, consider the case of *complete* unbinding which

occurs for a repulsive total interaction,  $V_{TI} \geq 0$ , as the effective pressure,  $H$ , goes to zero.<sup>60,1,61</sup> Then,

$$\bar{\ell} \sim H^{-\psi^c} \quad , \quad \xi_{\perp} \sim H^{-\nu_{\perp}^c} \quad , \quad \text{and} \quad \xi_{\parallel} \sim H^{-\nu_{\parallel}^c} \quad . \quad (3.23)$$

with critical exponents  $\psi^c$ ,  $\nu_{\perp}^c$ , and  $\nu_{\parallel}^c$ . (The superscript c stands for 'complete'). Likewise, the surface free energy,  $f_s(H) \equiv H\bar{\ell} + V_{TI}(\bar{\ell})$  with  $\bar{\ell} = \bar{\ell}(H)$  contains the singular part,  $f_s \sim H^{2-\alpha^c}$ , and

$$\partial^2 f_s / \partial H^2 = \partial \bar{\ell} / \partial H \sim H^{-\alpha^c} \quad . \quad (3.24)$$

The four critical exponents  $\psi^c$ ,  $\nu_{\perp}^c$ ,  $\nu_{\parallel}^c$ , and  $\alpha^c$  are not independent but satisfy the scaling relations

$$\alpha^c = \psi^c + 1 \quad , \quad \nu_{\perp}^c = \zeta \nu_{\parallel}^c \quad , \quad \text{and} \quad 2n\nu_{\parallel}^c = 2 - \alpha^c + 2\psi^c \quad (3.25)$$

with  $n$  as in (2.19). Therefore, there is *only one independent* critical exponent, say  $\psi^c$ .<sup>60</sup> For a direct interaction,  $V_{DI}(\ell) \sim 1/\ell^p$ , one has

$$\begin{aligned} \psi^c &= 1/(1+p) \quad \text{for} \quad p < \tau \\ &= 1/(1+\tau) \quad \text{for} \quad p \geq \tau. \end{aligned} \quad (3.26)$$

This holds both for thermally-excited<sup>60,61</sup> and for impurity-induced<sup>1,2,62</sup> fluctuations with  $\tau$  as given by (2.22) and (2.23), respectively.

For  $H = 0$ , the surfaces may be bound (in the thermodynamic limit,  $L_{\parallel} = \infty$ ), i.e., the total interaction,  $V_{TI}$ , may have an attractive part. The strength of this attraction depends on various parameters, and can vanish as these parameters are changed. Then, the surfaces undergo a continuous or discontinuous unbinding transition.

At a *discontinuous* transition, the mean separation,  $\bar{\ell}$ , jumps from a finite value to infinity.<sup>63</sup> Such a transition can occur for direct interactions which satisfy  $V_{DI}(\ell) \gg V_{FL}(\ell) \sim 1/\ell^{\tau}$  for large  $\ell$ , and which have an attractive part at smaller values of  $\ell$ . In contrast, a discontinuous transition is impossible in the SFL regime with  $|V_{DI}(\ell)| \ll V_{FL}(\ell) \sim 1/\ell^{\tau}$ .<sup>48,58</sup> In the IFL regime with  $|V_{DI}(\ell)| \sim V_{FL}(\ell)$ , discontinuous transitions are still possible but acquire very unusual scaling properties.<sup>48,49,64</sup>

A *continuous* transition is *critical* if the singular behavior depends on *two* scaling fields.<sup>65</sup> This applies to *all* transitions in the SFL regime.<sup>58</sup> At a critical unbinding transition, one has  $\bar{\ell} = y^{-\psi} \Omega_{\ell}(H/y^{\Delta})$  where  $y$  is an appropriate scaling field. Similar scaling forms with corresponding exponents  $\nu_{\perp}$ ,  $\nu_{\parallel}$ , and  $2 - \alpha$  hold for  $\xi_{\perp}$ ,  $\xi_{\parallel}$ , and  $f_s$ . For  $y = 0$ , one has  $\bar{\ell} \sim H^{-\psi/\Delta}$  and, thus,  $\psi^c = \psi/\Delta$  etc. It then follows from (3.25) that  $\nu_{\perp} = \zeta \nu_{\parallel}$  and  $2n\nu_{\parallel} = 2 - \alpha + 2\psi$ .

### 3.8 How far is it to asymptotia?

In the previous subsections, the *asymptotic* behavior associated with the unbinding of surfaces has been discussed. If one wants to study these critical effects

in experiments or numerical simulations, one must worry about: (i) *Crossover behavior*; (ii) *Finite size effects*; and (iii) *Time scales* for equilibration. The first two topics are discussed in this subsection while equilibration will be considered in Sec. 4.

**1. Crossover behavior** — As an example, consider complete wetting (or edge melting) in  $d = 2$  for a direct interaction,  $V_{DI}(\ell) \approx C_p/\ell^p$  with  $p > \tau = 2$ . Then, the unbinding of the 1-dim interfaces is driven by thermally-excited fluctuations for  $T > 0$ . Indeed, their mean separation behaves as<sup>60</sup>

$$\bar{\ell} \approx cs^{1/3}/h^{1/3} \quad \text{for small } h \equiv H/C_p \quad \text{with } s \equiv T^2/\tilde{\Sigma}C_p$$

and interfacial stiffness  $\tilde{\Sigma} = \tilde{\Sigma}(T)$ . If the 1-dim interface feels a periodic potential, one expects<sup>66</sup>  $\tilde{\Sigma} \sim \exp(J/T)$  and, thus,  $s \sim \exp(-J/T)$  for small  $T$  where  $J$  is the step energy. At  $T = 0$ , on the other hand, the unbinding is entirely controlled by  $V_{DI}$ , and  $\bar{\ell} \sim 1/h^{1/(p+1)}$  for small  $h$  as follows from minimization of  $V(\ell) \approx H\ell + C_p/\ell^p$ .

Thus, one has a characteristic *crossover* at low  $T$  which can be described by the scaling form

$$\bar{\ell} = h^{-1/(1+p)} \Omega(s/h^\phi) \quad \text{with } \phi = (p-2)/(p+1) \quad (3.27)$$

where the shape function  $\Omega(x) \approx cx^{1/3}$  for large  $x$ . It seems that such a crossover has been observed, for  $p = 3$ , in recent experiments<sup>31</sup> on edge melting.

**2. Finite size effects** — Now, assume that the surfaces are embedded in a finite system which has a linear extension,  $N_{\parallel}$  and  $N_{\perp}$ , in the direction parallel and perpendicular to the surfaces. The anisotropy of the surface fluctuations, characterized by  $\xi_{\perp} \sim \xi_{\parallel}^{\zeta}$  leads to different finite size effects in these two directions.

In the parallel direction, finite size effects set in once  $\xi_{\parallel} \simeq N_{\parallel}$ . This is very important for the interpretation of computer simulations. It leads, in fact, to quasicritical behavior as  $H \rightarrow 0$  for finite  $N_{\parallel}$ .<sup>67</sup> In the perpendicular direction, finite size effects are always present for  $\xi_{\perp} \simeq N_{\perp}$  since the boundaries then act as a confining potential on the surfaces. However, such effects may set in even for  $\xi_{\perp} \ll N_{\perp}$ . This happens in the context of wetting. In this case, a finite value of  $N_{\perp}$  leads to a shift of the coexistence curve and the process of wetting is truncated at  $H = H_* > 0$ . Quite generally, one has  $H_* \sim 1/N_{\perp}$ .<sup>68</sup> Since the limit  $H = 0$  is no longer accessible, the mean separation  $\bar{\ell}$  of the surfaces can no longer diverge. For a non-conserved order parameter, the maximal value of the equilibrium separation is given by<sup>68</sup>

$$\max(\bar{\ell}) \sim 1/H_*^{\psi^c} \sim N_{\perp}^{\psi^c} \quad . \quad (3.28)$$

with  $\psi^c$  as in (3.26). This behavior applies, e.g., to wetting by one-component fluids (where it is called capillary condensation) and to surface melting. For a conserved order parameter, the truncation of  $\bar{\ell}$  sets in somewhat earlier, and  $\max(\bar{\ell}) \sim N_{\perp}^{\psi^c/(1+\psi^c)}$ .

## 4 Interface Dynamics and Growth of Wetting Layers

### 4.1 Introduction

In *equilibrium*, interacting surfaces feel two forces: an external pressure,  $H$ , which is balanced, for  $H \geq 0$ , by the disjoining pressure,  $-\partial V_{TI}/\partial \ell$ , arising from the interactions.<sup>40</sup> Then, the total force,

$$\hat{H}(\ell) \equiv H + \partial V_{TI}/\partial \ell \quad (4.1)$$

acting on the surfaces, vanishes as in (3.1). Now, consider an *unbalanced* situation *away* from equilibrium with  $\hat{H}(\ell) \neq 0$ . Then, the surfaces will move apart for  $\hat{H}(\ell) < 0$  or will come closer together for  $\hat{H}(\ell) > 0$ .

In the following, I will focus on the *thickening of wetting layers* or on the *growth of thin films*. In this case, the total force,  $\hat{H} < 0$ , corresponds to an effective *undersaturation*. This undersaturation *scales* as

$$\hat{H} \sim \partial V_{TI}/\partial \ell \sim -1/\ell^{1+\psi^c} \quad \text{for } H = 0 \quad (4.2)$$

with  $\psi^c = 1/(1+p)$  or  $\psi^c = 1/(1+\tau)$  as in (3.26). Thus, for  $H = 0$ , the driving force for the dynamics is either determined by the direct repulsion  $V_{DR} \sim 1/\ell^p$  or by the fluctuation-induced repulsion  $V_{FL} \sim 1/\xi_{\perp}^{\tau}$ .

Several growth modes for the thickening of wetting layers will be discussed: (i) Adhesive growth for rough interfaces;<sup>69</sup> (ii) Adhesive growth for smooth interfaces; (iii) Diffusion-limited growth;<sup>70</sup> and (iv) Activated growth in the presence of quenched impurities. In all cases, the wetting layers are taken to be close to thermal and *chemical* equilibrium. This has to be distinguished from the dynamics of dry spreading<sup>71</sup> where the total volume of the wetting film is fixed.

### 4.2 Adhesive growth for rough interfaces

Consider a liquid phase,  $\beta$ , which is adsorbed from a vapor phase,  $\alpha$ , onto a solid substrate,  $\gamma$ . Then, a wetting layer builds up and the mean separation,  $\bar{\ell}$ , of the  $(\alpha\beta)$  interface from the solid wall steadily increases with time,  $t$ . The deposition rate from the vapor phase is proportional to the vapor pressure while the evaporation rate depends on the binding energies within the condensed phase. The growth rate is then proportional to the undersaturation:  $\partial \bar{\ell}/\partial t \sim -\hat{H}(\bar{\ell})$ , which implies<sup>69</sup>

$$\bar{\ell}(t) \sim t^{\theta} \quad \text{with } \theta = \psi^c/(1+\psi^c) \quad \text{for } H = 0 \quad (4.3)$$

where  $\psi^c$  is given by (3.26) and  $\tau = (d-1)/\zeta$ . On the other hand, for small  $H > 0$ , the equilibrium thickness  $\bar{\ell}(\infty) \sim 1/H^{\psi^c}$ . It then follows that the equilibration time,  $t_{eq}$ , scales as  $t_{eq} \sim 1/H^{1+\psi^c}$  for small  $H > 0$ . For a marginally rough

interface in  $d = 3$  and  $V_{DI}(\ell) \sim \exp(-\ell/a_1)$ , the mean separation  $\bar{\ell}(t) \sim \ln(t)$ , and the equilibration time  $t_{eq} \sim 1/H$ .<sup>72</sup>

This growth mode also applies to wetting in lattice models with nonconserved dynamics provided  $T > T_R^{\alpha\beta}$  where  $T_R^{\alpha\beta}$  is the roughening temperature of the  $(\alpha\beta)$  interface. Indeed, the growth law (4.3) has been confirmed by MC simulations of SOS-models in  $d = 2$  and  $d = 3$ <sup>73</sup>, of a 3-state chiral Potts model in  $d = 2$ <sup>74</sup>, and of an Ising model in  $d = 3$ <sup>75</sup>. For a 2-dim model on a periodic lattice (with short-ranged interactions), the growth law (4.3) becomes  $\bar{\ell}(t) \approx A_\ell t^{1/4}$  with amplitude  $A_\ell \sim (T^2/\tilde{\Sigma})^{1/4} \sim \exp(-J/T)$  since the stiffness  $\tilde{\Sigma} \sim \exp(J/T)$  for low  $T$ . Therefore,  $\bar{\ell}(t)$  exhibits strong crossover behavior at low  $T$  which has indeed been observed in one of the MC studies.

What about the length scales  $\xi_{\parallel}$  and  $\xi_{\perp}$ ? Scaling implies that  $\xi_{\parallel}(t) \approx b\xi_{\parallel}(b^{-z}t)$  and, thus,<sup>69</sup>

$$\xi_{\parallel}(t) \sim t^{\theta_{\parallel}} \quad \text{with} \quad \theta_{\parallel} = 1/z \quad (4.4)$$

which will be taken as a *definition* of the dynamic critical exponent,  $z$ . On the other hand, one may again define a roughness exponent,  $\zeta$ , via  $\xi_{\perp}(t) \sim \xi_{\parallel}(t)^{\zeta}$  as before. Then,

$$\xi_{\perp}(t) \sim t^{\theta_{\perp}} \quad \text{with} \quad \theta_{\perp} = \zeta/z. \quad (4.5)$$

The growth of  $\bar{\ell}$  as given by (4.3) is rather slow. Then, one may assume *local equilibrium* such that the length scales  $\bar{\ell}$ ,  $\xi_{\parallel}$ , and  $\xi_{\perp}$  are related via  $\xi_{\parallel} \sim \bar{\ell}^{\nu_{\parallel}/\psi}$  and  $\xi_{\perp} \sim \xi_{\parallel}^{\zeta}$  where the exponents  $\nu_{\parallel}/\psi = \nu_{\parallel}^c/\psi^c$  and  $\zeta$  have the same values as in equilibrium. This implies that  $\theta_{\parallel} = \theta_{\parallel}^c/\psi^c = 1/(2 - \eta_{\sigma})$ ,  $\theta_{\perp} = \zeta/(2 - \eta_{\sigma})$  and  $z = 2 - \eta_{\sigma}$  with  $\eta_{\sigma} \geq 0$ . The local equilibrium assumption is indeed confirmed by a systematic study of the Langevin equation  $\partial\ell/\partial t = -C\delta\mathcal{H}/\delta\ell + f$  with effective Hamiltonian  $\mathcal{H}\{\ell\}$  as in (3.10) and Gaussian white noise,  $f$ .<sup>69</sup>

It is possible, however, that the value of  $\zeta$  is determined by the *dynamics* rather than by the statics. Such a behavior has been found for a *single* interface which grows by *ballistic deposition* with a local growth rule.<sup>76,77</sup> For some growth rules, the interfacial coordinate,  $\ell(x, t) = gt + h(x, t)$ , evolves according to<sup>76</sup>  $\partial h/\partial t = g(\nabla h)^2 + CK\nabla^2\ell + f$ . This leads to  $z = z_1$  and  $\zeta = \zeta_1$  with  $z_1 + \zeta_1 = 2$ .<sup>78</sup> In  $d = 1 + 1$ , one has  $z_1 = 3/2$  and  $\zeta_1 = 1/2$ .<sup>76</sup> For  $d > 2$ , the exponents are expected to satisfy the bounds  $1/d \leq \zeta_1 \leq 1/2$  and  $3/2 \leq z_1 \leq 2 - 1/d$ .<sup>76,79</sup>

In the present context, the velocity,  $g$ , is not constant but depends on time:  $g = \partial\ell/\partial t \sim t^{\theta-1}$ . Therefore, the fluctuations,  $h(\vec{x}, t) = \ell(\vec{x}, t) - \bar{\ell}(t)$ , should evolve according to

$$\partial h/\partial t = Bt^{\theta-1}(\nabla h)^2 + CK\nabla^{2n}\ell + f. \quad (4.6)$$

where  $f$  is again a Gaussian white noise. A scaling analysis of this equation shows that the nonlinear term  $\sim (\nabla h)^2$  is *irrelevant* for  $\theta < \theta_* \equiv z_1/2n$ . The adhesive growth law (4.3) implies  $\theta < 1/2$  for  $\psi^c < 1$ , while  $\theta_* \geq 3/4$  follows from the bound  $z_1 \geq 3/2$ . Thus, local equilibrium should be generally valid for (4.3).

On the other hand, if  $\theta > \theta_*$  is enforced by  $H < 0$  or by an external potential, one may enter the ballistic deposition regime which is then characterized by

$\zeta = \zeta_1$  and  $z = z_1/\theta$ . Even in this regime,  $\xi_\perp$  is always small compared to  $\bar{\ell}$  since  $\theta_\perp = \zeta/z = \theta\zeta_1/(2 - \zeta_1) \leq \theta/3$ .

### 4.3 Adhesive growth for smooth interfaces

Now, consider a wetting layer in  $d = 3$  consisting of a periodic crystal or an ideal quasicrystal. At low T, the interfaces bounding this layer are *smooth* (with  $\zeta = 0$ ), and the approach to complete wetting proceeds via multilayering. Each new layer starts from 2-dim nucleation clusters with critical radius  $\simeq \Sigma_s/a(-\hat{H})$  and free energy  $\Delta F \simeq \Sigma_s^2/a(-\hat{H})$  where  $\Sigma_s$  is the step free energy per unit length. Therefore, the growth is activated, and  $\partial\ell/\partial t \sim \exp(-\Delta F/T) \sim \exp[C/\hat{H}(\ell)]$ . For  $H = 0$  and  $V_{DI} \sim 1/\ell^p$ , this leads to the logarithmic growth law

$$\bar{\ell}(t) \sim [\ln(t)]^{\psi^c} \quad \text{with} \quad \psi^c = 1/(1+p) \quad (4.7)$$

in  $d = 3$ . For small  $H > 0$ , the equilibrium separation  $\bar{\ell}(\infty) \sim 1/H^{\psi^c}$  which implies the equilibration time  $t_{eq} \sim \exp[C/H]$  for small  $H > 0$ .

The above analysis applies to multilayering in the 3-dim Ising model as has been studied in a MC simulation.<sup>54</sup> In this case,  $V_{TI}(\ell) \sim \exp(-c\ell/a)$  which implies  $\bar{\ell}(t) \sim \ln[\ln(t)]$  for  $H = 0$ , and  $t_{eq} \sim \exp[C/H]$  as before. Thus, observation of more than the first few layers requires an exponentially large time.

In real solid films, *defects* have a dramatic effect on the growth rate. In epitaxial growth, *mismatch dislocations* lead to an effective interaction,  $V_{DI}(\ell) \sim -1/\ell$ , which is *attractive* and, thus, prevents complete wetting at  $H = 0$ .<sup>80</sup> On the other hand, for non-epitaxial growth, the growth rate can be greatly enhanced by the presence of *screw dislocations* which act as a source for steps. Classical theories for spiral growth predict that the rate is  $\sim$  (supersaturation)<sup>2</sup>.<sup>81</sup> In the present context, this leads to  $\partial\ell/\partial t \sim [\hat{H}(\ell)]^2$ . For  $H = 0$  and  $V_{DI}(\ell) \sim 1/\ell^p$ , this implies the power law growth

$$\bar{\ell}(t) \sim t^\theta \quad \text{with} \quad \theta = \psi^c/(\psi^c + 2) \quad \text{and} \quad \psi^c = 1/(1+p) \quad (4.8)$$

in  $d = 3$ .

### 4.4 Diffusion-limited growth

Next, consider a binary mixture or alloy of two molecular species, A and B, which separates into two phases,  $\alpha$  and  $\beta$ , below its consolute point. Assume that the B-rich phase  $\beta$  forms a wetting layer which intrudes between the  $\alpha$  phase and a solid wall,  $\gamma$ . At complete wetting, the  $\beta$  layer will grow into the  $\alpha$  phase, and the region of  $\alpha$  phase adjacent to the  $(\alpha\beta)$  interface becomes depleted of B molecules. If there is no hydrodynamic flow, further growth can only occur by diffusion through the depleted region.

The thickness of the depleted region is set by the diffusion length,  $\delta(t) \sim t^{1/2}$ . Mass conservation implies that  $\ell(t) \sim [X_\alpha - X(\ell)]\delta(t)$  where  $X_\alpha$  is the concentration of B molecules deep in the  $\alpha$  phase while  $X(\ell)$  is the concentration in front

of the  $(\alpha\beta)$  interface. In local equilibrium, one has  $X_\alpha - X(\ell) \sim$  undersaturation of the relative chemical potential. Then, the thickness,  $\ell$  of the wetting layer evolves according to  $\ell(t) \sim -\hat{H}(\ell)/\delta(t)$ , which leads to<sup>70</sup>

$$\bar{\ell}(t) \sim t^\theta \quad \text{with} \quad \theta = \psi^c/2(1 + \psi^c) \quad . \quad (4.9)$$

where  $\psi^c$  is given by (3.26) with  $\tau = (d-1)/\zeta$ . This is confirmed by a systematic study of the interfacial motion using the Green's function formalism.<sup>70</sup>

An interface which moves as a result of bulk diffusion can be unstable with respect to the Mullins–Sekerka instability.<sup>82</sup> It turns out, however, that the interfacial motion is *stable* as long as the growth exponent  $\theta < 1/2$  which applies to the growth given by (4.9).<sup>70</sup> On the other hand, for effective pressure  $H < 0$ , the above scaling analysis leads to  $\bar{\ell}(t) \sim \delta(t) \sim t^{1/2}$ , and the interface develops 'fingers' as a result of the Mullins–Sekerka instability. This fingering could accelerate the interfacial motion and  $\theta$  could become larger than  $1/2$ .

## 4.5 Activated growth in the presence of quenched impurities

Finally, consider the same wetting geometry as in the previous subsections but assume that the  $(\alpha\beta)$  interface feels a random potential arising from the presence of quenched impurities. In such a situation, the interface gets caught in *metastable* states and its dynamics is then controlled by the size of activation barriers.<sup>83,13</sup>

In order to make an interfacial fluctuation of longitudinal and transverse extension,  $\xi_{\parallel}$  and  $\xi_{\perp} \sim \xi_{\parallel}^{\zeta}$ , one must overcome free energy barriers,  $\Delta F \sim \xi_{\parallel}^{d-1}(\xi_{\perp}/\xi_{\parallel})^2$ . On the other hand, if the interface moves out by such a hump, it will typically gain a free energy  $\sim (-\hat{H})\xi_{\perp}\xi_{\parallel}^{d-1}$ . Therefore, the interface must nucleate humps which are characterized by a critical size  $\xi_{\parallel} = \xi_{\parallel c} \sim (-\hat{H})^{-1/(2-\zeta)}$  corresponding to an activation free energy  $\Delta F \sim (-\hat{H})^{-(d-3+2\zeta)/(2-\zeta)}$ . This leads to  $\partial\ell/\partial t \sim \exp[-\Delta F/T] \sim \exp[-C/(-\hat{H}(\ell))^{\chi}]$ , and

$$\bar{\ell}(t) \sim [\ln(t)]^{\psi^c/\chi} \quad \text{with} \quad \chi = (d-3+2\zeta)/(2-\zeta) \quad (4.10)$$

where  $\psi^c$  is given by (3.26) with  $\tau = 2(1/\zeta - 1)$ .

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