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Typical and Exceptional Shape Fluctuations of Interacting Strings.

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Abstract. - 1-dimensional strings governed by a finite tension are studied in $d = 1 + d_\perp$ dimensions. In the presence of a short-ranged attractive interaction, two strings undergo a nontrivial unbinding transition for any value of $d_\perp$. The characteristic unbinding temperature T_* vanishes as $1/d_\perp$ for large $d_\perp$. The critical exponents evolve with $d_\perp$ in a singular and nonmonotonic way. For $d_\perp < 4$, the typical string fluctuations consist of humps with roughness exponent $\zeta = 1/2$; for $d_\perp > 4$, on the other hand, these humps become more and more exceptional as the transition is approached and are then governed by a power law distribution.

Consider a $d_\parallel$-dimensional manifold embedded in $d = d_\parallel + d_\perp$ dimensions. The shape fluctuations of such a manifold lead to a variety of critical phenomena [1]. Examples are provided i) by the thermally-excited roughness of polymerized membranes in $d = 2 + d_\perp$ [2-4]; ii) by the roughness of strings (or directed polymers) in $d = 1 + d_\perp$ when subject to frozen randomness [5,6], and iii) by unbinding transitions of interfaces and membranes in $d = d_\parallel + 1$, which represent wetting and adhesion transitions, respectively [7]. These critical effects have been recently studied by a variety of theoretical methods. The results of these studies are all consistent with the general expectation that the critical exponents which govern these effects vary with d in a *smooth* and *monotonic* fashion.

In this paper, I study two interacting strings in $d = 1 + d_\perp$ dimensions. Here and below, the term «string» is used for a 1-dimensional object which has two properties: i) it is directed in the sense that its tangent vectors point, on average, into a certain direction, and ii) its shape fluctuations are effectively governed by a finite tension. Physical examples are domain walls in adsorbed monolayers, vortex lines in type-II superconductors, directed cylindrical micelles in chemical equilibrium with the surrounding medium, and presumably some directed polymers [8] such as polyelectrolytes.

For any value of $d = 1 + d_\perp$, two strings are found to undergo a nontrivial unbinding transition when they interact with a short-ranged attractive potential. The critical behaviour at these transitions can be determined exactly by transfer matrix methods and is found to have the following properties: i) the critical exponents do *not* vary smoothly with d. For example, the critical exponent $\nu_\parallel$ for the correlation length is a piecewise parabolic function of $d_\perp$, see eq. (17) below; ii) the exponents do *not* vary monotonically with $d_\perp$, see again eq. (17); iii) the character of the shape fluctuations is qualitatively different for $d_\perp < 4$ and for $d_\perp > 4$. For $d_\perp < 4$, the *typical* string fluctuations consist of an ensemble of humps, which are characterized by the roughness exponent $\zeta = 1/2$; for $d_\perp > 4$, on the other hand, these humps become more and more *exceptional* as the transition is approached.

A priori, one might expect that $d_\perp = 0$ and $d_\perp = \infty$ represent borderline dimensionalities for the critical behaviour at unbinding transitions in $d = 1 + d_\perp$. This expectation is indeed confirmed by the exact results reported here. However, the complex evolution of the critical behaviour with d implies i) that a perturbative expansion around $d_\perp = 0$ can, in general, only be trusted for $d_\perp \leq 1$, and ii) that a perturbative expansion around $d_\perp = \infty$ in powers of $1/d_\perp$ must fail completely, see eq. (23) below.

In the language of the renormalization group, the behaviour found here arises from the presence of a *marginal* «operator» the strength of which depends on $d_\perp$. This «operator» represents an effective potential between the strings. More precisely, this potential acts on the *absolute value* of the relative displacement, $\boldsymbol{z}$, of the strings (for $d_\perp \neq 1$) and arises from an average over the *angular* fluctuations of this displacement. Scaling arguments indicate that such a marginal «operator» arising from the angular fluctuations should, in general, be present for unbinding transitions in $d = d_\parallel + d_\perp$ provided $d_\perp \neq 1$.

To proceed, consider two strings (or directed polymers or vortex lines) which are, on average, parallel. Their position is measured by the coordinates $\boldsymbol{z}_a(x)$ and $\boldsymbol{z}_b(x)$, respectively, where $\boldsymbol{z}$ is a $d_\perp$-dimensional vector. Their tensions are denoted by K_a and K_b, and their mutual interaction by $V(|\boldsymbol{z}_a - \boldsymbol{z}_b|)$ which is taken to depend only on their local distance. The effective Hamiltonian for the two strings is then given by

$$\mathscr{H}^{ab}\{\boldsymbol{z}_a, \boldsymbol{z}_b\} = \int \mathrm{d}x \left\{ \frac{1}{2} K_a (\mathrm{d}\boldsymbol{z}_a/\mathrm{d}x)^2 + \frac{1}{2} K_b (\mathrm{d}\boldsymbol{z}_b/\mathrm{d}x)^2 + V(|\boldsymbol{z}_a - \boldsymbol{z}_b|) \right\}. \tag{1}$$

As usual, one may separate this Hamiltonian into two parts depending on the «centre-of-mass» coordinate, $\boldsymbol{z}_0 \equiv (K_a \boldsymbol{z}_a + K_b \boldsymbol{z}_b)/(K_a + K_b)$, and on the relative coordinate, $\boldsymbol{z} \equiv \boldsymbol{z}_b - \boldsymbol{z}_a$ [4]. The latter coordinate is then governed by

$$\mathscr{H}\{\boldsymbol{z}\} = \int \mathrm{d}x \left\{ \frac{1}{2} K (\mathrm{d}\boldsymbol{z}/\mathrm{d}x)^2 + V(|\boldsymbol{z}|) \right\} \quad \text{with} \quad K \equiv K_a K_b/(K_a + K_b). \tag{2}$$

In the following, I will focus on interaction potentials $V(z)$ with $z \equiv |\boldsymbol{z}|$ which are *short-ranged.* More precisely, I will assume i) that $V(z)$ has a hard wall at $z = a > 0$ with $V(z) = \infty$ for $z < a$; ii) that $V(z)$ has a finite range, $z = z_0 > a$, with $V(z) = 0$ for $z > z_0$; and iii) that $V(z)$ is attractive, *i.e.* that $V(z) < 0$ for $a < z < z_0$.

Since x represents a 1-dimensional coordinate (which plays the role of «time»), the statistical properties of the model (2) can be studied by transfer matrix methods which lead to the $d_\perp$-dimensional Schrödinger-type equation

$$\{-(T^2/2K)\nabla_\perp^2 + V(|\boldsymbol{z}|)\}\, \Psi_n(\boldsymbol{z}) = E_n \Psi_n(\boldsymbol{z}) \tag{3}$$

with the Laplacian $\nabla_\perp^2 \equiv \sum_{\alpha=1}^{d_\perp} \partial^2/\partial z_\alpha^2$.

Since the interaction potential V depends only on $z = |\boldsymbol{z}|$, the Laplacian may be decomposed into its radial and angular part [9]:

$$\nabla_\perp^2 = \frac{\partial^2}{\partial z^2} + \frac{d_\perp - 1}{z} \frac{\partial}{\partial z} + \frac{\hat{A}}{z^2}. \tag{4}$$

The eigenvalues of the angular-momentum operator, $\hat{A}$, are $-l(l+d_\perp-2)$ with integer $l=0,1,2,\ldots$. The term $\sim\partial/\partial z$ in (4) can be eliminated by the transformation $\phi_n(z)\equiv \equiv z^{(d_\perp-1)/2}\psi_n(z)$, where ψ_n represents the radial part of $\Psi_n(z)$. In addition, it is convenient to introduce the dimensionless coordinate $y\equiv z/z_0$, the rescaled potential $\bar{V}(y)\equiv \equiv 2Kz_0^2V(z_0y)/T^2$, and the rescaled energies $\bar{E}_n\equiv 2Kz_0^2E_n/T^2$. One then arrives at

$$\left\{-\frac{\partial^2}{\partial y^2}+\frac{A_l(d_\perp)}{y^2}+\bar{V}(y)\right\}\phi_n(y)=\bar{E}_n\phi_n(y) \tag{5}$$

with

$$A_l(d_\perp)\equiv\frac{1}{4}(d_\perp-3+2l)(d_\perp-1+2l). \tag{6}$$

The range of $\bar{V}(y)$ is given by $y_0=1$. Note that the total potential now consists of the short-ranged part, $\bar{V}(y)$, which is proportional to $1/T^2$, and the inverse power law potential, $A_l(d_\perp)/y^2$, which is independent of temperature. Such a potential belongs to the so-called intermediate fluctuation regime [10,11].

Within the transfer matrix approach, the statistical properties of the fluctuating field $y\sim z$ can be obtained from the ground state, $\phi_0(y)$, and the ground-state energy $\bar{E}_0$ In particular, the interacting strings undergo an unbinding transition when $\bar{E}_0$ goes to zero from below and merges with the band edge of scattering states. In order to determine the ground-state properties, one has to consider the Schrödinger-type equation (5) with $A_l(d_\perp)$ replaced by

$$A(d_\perp)\equiv\min_l[A_l(d_\perp)]=\begin{cases}A_0(d_\perp) & \text{for } d_\perp\geq 1,\\ A_l(d_\perp) & \text{for } 3-2l>d_\perp\geq 1-2l.\end{cases} \tag{7}$$

Thus, the amplitude $A(d_\perp)$ of the power law potential $\sim 1/y^2$ is a piecewise parabolic function of $d_\perp$, which is periodic for $d_\perp<3$.

Furthermore, one has the relation $A(d_\perp)\geq -1/4$. It then follows that the ground-state energy $\bar{E}_0$ vanishes at a certain strength of the short-ranged attraction $\bar{V}(y)$ [10]. Thus, the interacting strings undergo an unbinding transition for any value of $d_\perp$. In what follows, I will focus on the physical region $d_\perp\geq 0$.

For $\bar{E}_0=0$, the total potential $V_T(y)\equiv\bar{V}(y)+A(d_\perp)/y^2$ has a local minimum at $y=y_m<1$, where $V_T(y_m)$ is negative and of order one [10]. For large $d_\perp$, the amplitude $A(d_\perp)$ as given by (6) and (7) behaves as $A(d_\perp)\approx d_\perp^2/4$. Since $\bar{V}(y)$ is proportional to $1/T^2$, the unbinding temperature, $T=T_*$, scales as

$$T_*\sim 1/d_\perp \quad \text{for large } d_\perp. \tag{8}$$

For $y\geq 1$, the short-ranged interaction potential vanishes, and the ground state $\phi_0(y)$ of (5)-(7) is given by

$$\phi_0^>(y)\propto\sqrt{y}K_\mu(qy) \quad \text{with } q\equiv\sqrt{|\bar{E}_0|}, \tag{9}$$

where $K_\mu(t)$ is a modified Bessel function with index

$$\mu=\left[\frac{1}{4}+A(d_\perp)\right]^{1/2}=\begin{cases}\frac{1}{2}d_\perp & \text{for } 0\leq d_\perp<1,\\ \frac{1}{2}|d_\perp-2| & \text{for } 1<d_\perp.\end{cases} \tag{10}$$

For $y < 1$, the form of ϕ_0 is determined by the short-ranged potential, $\bar{V}(y)$. The two pieces of ϕ_0 must match at $y = 1$. This implies

$$\left.\frac{\mathrm{d}\phi_0^{>}(y)}{\mathrm{d}y}\right|_{y=1} = C_1 \phi_0^{>}(y)|_{y=1} \quad \text{with } C_1 = C_1(T) \tag{11}$$

which represents an effective boundary condition for $\phi_0^{>}(y)$. Inserting the explicit form (9) into (11), one obtains

$$C_1(T) = \frac{1}{2} + q\frac{\partial K_\mu(q)}{\partial q}\Big/K_\mu(q) \equiv \frac{1}{2} - \mu + X_\mu(q). \tag{12}$$

The asymptotic behaviour of the modified Bessel function $K_\mu(q)$ for small q implies that $X_\mu(q=0) = 0$.

The unbinding temperature, T_*, is now implicitly given by

$$C_1(T_*) = \begin{cases} \frac{1}{2} - \mu = \frac{1}{2}(1 - d_\perp) & \text{for } 0 \leqslant d_\perp \leqslant 1, \\ \frac{1}{2}\,[1 - |d_\perp - 2|] & \text{for } 1 < d_\perp. \end{cases} \tag{13}$$

Note that $C_1(T_*)$ is nonmonotonic for $0 \leqslant d_\perp \leqslant 3$. The critical behaviour of q then follows from the expansion $C_1(T) \approx C_1(T_*) + C_1'(T - T_*)$ which implies

$$C_1'(T - T_*) \approx X_\mu(q) \quad \text{for small } |T - T_*|. \tag{14}$$

The variable $q = \sqrt{|\bar{E}_0|}$ is directly related to the longitudinal correlation length $\xi_\parallel$ via

$$\xi_\parallel = T/|E_0| \sim 1/|\bar{E}_0| = 1/q^2. \tag{15}$$

The critical behaviour of this quantity is obtained when the asymptotic behaviour of $X_\mu(q)$ for small q is inserted into (14). As a result, one finds

$$\xi_\parallel \sim |T - T_*|^{-\nu_\parallel} \tag{16}$$

with

$$\nu_\parallel = \begin{cases} 2/d_\perp & \text{for } 0 \leqslant d_\perp < 1, \\ 2/|d_\perp - 2| & \text{for } 1 \leqslant d_\perp < 4, \\ 1 & \text{for } 4 \leqslant d_\perp. \end{cases} \tag{17}$$

For $d_\perp = 4$, $\xi_\parallel$ exhibits a confluent logarithmic singularity: $\xi_\parallel \sim \ln[|T - T_*|/T_*]/|T - T_*|$ For $d_\perp = 0$ and $d_\perp = 2$, the expression in (17) leads to $\nu_\parallel = \infty$ which corresponds to the essential singularity

$$\xi_\parallel \sim \exp[C/|T - T_*|], \tag{18}$$

where C is a nonuniversal constant as follows from the asymptotic behaviour $X_0(q) \sim 1/\ln(q)$ for small q.

The behaviour of the transverse fluctuations is contained in the probability distribution

$$P(y)=\phi_0^2(y)=yK_\mu^2(qy)\Big/\int_1^\infty \mathrm{d}y\, yK_\mu^2(qy). \tag{19}$$

Close to the unbinding transition, *i.e.* for small q, this distribution behaves as

$$P(y)\approx\begin{cases} C_< q^2 yK_\mu^2(qy) & \text{for } 0\leq d_\perp<4, \\ C_> q^{d_\perp-2} yK_{(d_\perp-2)/2}^2(qy) & \text{for } 4<d_\perp, \end{cases} \tag{20}$$

where the index μ is given by (10). For $d_\perp=4$, one has $P(y)\sim q^2 yK_1^2(qy)/\ln(1/q)$

These distribution functions determine the critical behaviour of the moments, $\langle y^n\rangle$. A straightforward calculation leads to

$$\langle y^n\rangle\sim 1/q^n\sim\xi_\parallel^{n/2} \quad \text{for } d_\perp<4. \tag{21}$$

For $d_\perp>4$, on the other hand, one finds

$$\langle y^n\rangle\sim\begin{cases} q^{-n+d_\perp-4}\sim\xi_\parallel^{(n+4-d_\perp)/2} & \text{for } n>d_\perp-4, \\ \text{const} & \text{for } n<d_\perp-4. \end{cases} \tag{22}$$

In the latter case, the distribution attains the limiting form

$$P_*(y)=y^{3-d_\perp}/(d_\perp-4) \quad \text{at } T=T_*. \tag{23}$$

The distinct behaviour of the fluctuations for $d_\perp<4$ and $d_\perp>4$ can be understood in an intuitive way using the heuristic hump picture which has been previously developed for interfaces and membranes [1]. Thus, assume that the string fluctuations consist of humps with a lateral extension $\xi_\parallel$ and a transverse extension $1/q\sim\xi_\parallel^\zeta$ with the roughness exponent $\zeta=1/2$. The probability to find such a hump can be estimated by

$$\bar{P}(\xi_\parallel^\zeta)\equiv\int_{y_1}^{y_2}\mathrm{d}y\,P(y) \quad \text{with } y_1\equiv 1/bq \text{ and } y_2\equiv b/q, \tag{24}$$

where $b>1$ represents a scale factor. It follows from (20) that

$$\bar{P}(\xi_\parallel^\zeta)\sim\begin{cases} \text{const} & \text{for } d_\perp<4, \\ \xi_\parallel^{(4-d_\perp)\zeta} & \text{for } d_\perp>4, \end{cases} \tag{25}$$

in the limit of large $\xi_\parallel$ (for any value of $b>1$). The moments of y can now be estimated according to

$$\langle y^n\rangle\sim\xi_\parallel^{\zeta n}\bar{P}(\xi_\parallel^\zeta), \tag{26}$$

which again leads to the critical behaviour as given by (21) and (22).

Thus, the string fluctuations are indeed described by an ensemble of $\xi_\parallel$-humps with roughness exponent $\zeta=1/2$ for all values of $d_\perp$. However, the behaviour of $\bar{P}(\xi_\parallel^\zeta)$ as given by (25) implies that these humps represent *typical* fluctuations for $d_\perp<4$ but *exceptional* fluctuations for $d_\perp>4$.

REFERENCES

[1] For recent reviews, see LIPOWSKY R., in *Fundamental Problems in Statistical Mechanics VII,* edited by H. VAN BEIJEREN (North-Holland, Amsterdam) 1990; *Phys. Scr. T,* **29** (1989) 259.
[2] NELSON D. R. and PELITI L., *J. Phys. (Paris),* **48** (1987) 1085.
[3] DAVID F. and GUITTER E., *Europhys. Lett.,* **5** (1988) 709.
[4] LIPOWSKY R., *Europhys. Lett.,* **7** (1988) 255; LIPOWSKY R. and GIRARDET M., *Phys. Rev. Lett.,* **65** (1990) 2893.
[5] COOK J. and DERRIDA B., *Europhys. Lett.,* **10** (1989) 195.
[6] This problem is intimately related to the kinetic roughening of interfaces as reviewed by KRUG J. and SPOHN H., in *Solids Far from Equilibrium: Growth, Morphology and Defects,* edited by C. GODRÈCHE (Cambridge University Press) 1990; WOLF D. E., in *Kinetics of Ordering and Growth at Surfaces,* edited by M. LAGALLY (Plenum Press, New York, N.Y.) 1990.
[7] LIPOWSKY R., *Phys. Rev. Lett.,* **62** (1989) 704; GROTEHANS S. and LIPOWSKY R., *Phys. Rev. A,* **41** (1990) 4574.
[8] KANTOR Y. and KARDAR M., *Europhys. Lett.,* **9** (1989) 53.
[9] BALIAN R. and TOULOUSE G., *Ann. Phys. (N.Y.),* **83** (1974) 28.
[10] LIPOWSKY R. and NIEUWENHUIZEN T., *J. Phys. A,* **21** (1988) L-89; JÜLICHER F., LIPOWSKY R. and MÜLLER-KRUMBHAAR H., *Europhys. Lett.,* **11** (1990) 657; SPOHN H., *Europhys. Lett.,* **14** (1991) 689.
[11] FISHER M. E. and GEL'FAND M., *J. Stat. Phys.,* **53** (1988) 175, have shown that the necklace model for three random walkers in $d = 1 + 1$ belongs to the same regime.