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Critical behavior of interacting manifolds ☆

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Low-dimensional manifolds which interact via attractive forces can undergo an unbinding transition from a bound state at low temperatures to an unbound state at high temperatures. Three model systems are considered for which the critical behavior at these transitions can be determined exactly: (i) two interfaces in spatial dimensionality $d=1+1$ interacting with power-law interactions; (ii) the necklace model for p interfaces in $d=1+1$; and (iii) two interacting strings in $d=1+d_\perp$. It is shown that the critical behavior is identical for all three models provided one uses an appropriate identification of the model parameters.

1. Introduction

Many physical phenomena such as, e.g., wetting [1] and adhesion [2] are governed by the mutual interaction of low-dimensional manifolds such as interfaces and membranes. Quite generally, thermally excited shape fluctuations of these objects *renormalize* their direct interaction arising from intermolecular forces [3]. This renormalization acts to increase the repulsive part of the interaction. At low temperatures, the shape fluctuations are weak and the renormalized interaction closely resembles the direct interaction. However, as the temperature, T, is increased the renormalization becomes more and more effective up to a characteristic unbinding temperature, $T=T_*$, at which the manifolds undergo a transition from a bound to an unbound state. For interfaces and membranes, these *unbinding transitions* represent wetting and adhesion transitions, respectively.

The critical behavior at these transitions involves several length scales: the mean separation, $\langle \ell \rangle$, of the manifolds, the roughness $\xi_\perp \equiv \langle (\ell - \langle \ell \rangle)^2 \rangle^{1/2}$, and the longitudinal correlation length $\xi_\parallel$, see fig. 1. In many systems, the mean separation $\langle \ell \rangle$ and the roughness $\xi_\perp$ exhibit the same divergence at the transition. In this case, the probability to find a hump as shown in fig. 1 is always of order unity, and these humps represent typical shape fluctuations. However, it will become clear further below that, in some systems, this probability becomes smaller and smaller as the transition is approached. In the latter situation, the hump shown in fig. 1 represents an exceptional fluctuation.

☆ Dedicated to Michael E. Fisher on the occasion of his 60th birthday.

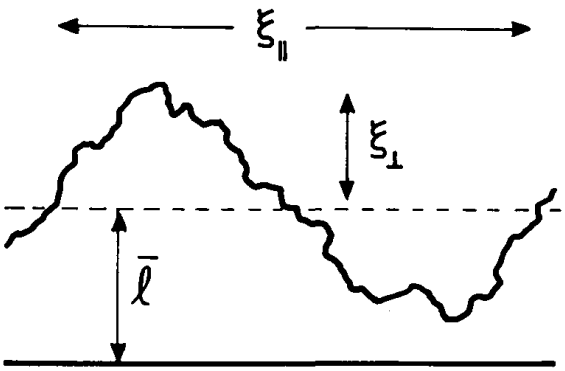

Fig. 1. Bound state of two interacting manifolds with mean separation $\bar{\ell} \equiv \langle \ell \rangle$, roughness $\xi_\perp \equiv \langle (\ell - \langle \ell \rangle)^2 \rangle^{1/2}$, and longitudinal correlation length $\xi_\parallel$.

In this short paper, I will focus on three systems for which the critical behavior associated with the unbinding transitions can be determined exactly: (i) two interfaces in $d=1+1$ interacting with a direct interaction $V(\ell) \sim 1/\ell^2$ [4-6]; (ii) the necklace model for p interfaces in $d=1+1$ [7-9]; and (iii) two interacting strings in $d=1+d_\perp$ [10]. In all cases, one finds that the critical behavior is nonuniversal and that the critical exponents depend on the parameters of the model. However, the critical singularities are identical for all three models provided one makes an appropriate identification of the model parameters, see relations (10) and (14) below.

2. Two interfaces or domain walls in $d=1+1$

First, let us consider two interfaces or domain walls in $d=1+1$ which (i) are, on average, parallel to a reference line with coordinate x, and (ii) have a finite interfacial tension. Their local separation, $\ell = \ell(x)$, is then governed by the effective Hamiltonian

$$\mathcal{H}\{\ell\} = \int \mathrm{d}x \left[\tfrac{1}{2} K \left(\frac{\mathrm{d}\ell}{\mathrm{d}x} \right)^2 + V[\ell(x)] \right], \tag{1}$$

where K is an appropriate interfacial stiffness and $V(\ell)$ represents the direct interaction. This interaction is taken to behave as

$$V(\ell) \approx -W/\ell^2 \qquad \text{for large } \ell\,. \tag{2}$$

The model as given by (1) and (2) can be studied by transfer matrix methods [11,4]. This leads to a one-dimensional Schrödinger-type equation for which the interaction $V(\ell)$ plays the role of a quantum-mechanical potential. It is convenient to consider the dimensionless interaction $v(z) \equiv 2K\ell_0^2 V(\ell_0 z)/T^2$, where ℓ_0 is a microscopic scale which characterizes the short-ranged part of $V(\ell)$. The asymptotic behavior in (2) then implies

$$v(z) \approx -w/z^2 \quad \text{for large } z \qquad \text{with } w \equiv 2KW/T^2\,, \tag{3}$$

irrespective of the choice of ℓ_0. Now, three cases must be distinguished: (i) for $w > \frac{1}{4}$, there is no unbinding transition since there is always an infinite number of bound states; (ii) As $w = \frac{1}{4}$ is approached from above, the two interfaces undergo an unbinding transition of infinite order provided the short-ranged part of $V(\ell)$ is not capable of binding the two interfaces; at this transition, the longitudinal correlation length, $\xi_\parallel$, behaves as [11]

$$\xi_\parallel \sim \exp(2\pi/\sqrt{w-\tfrac{1}{4}}) , \tag{4}$$

and the mean separation ℓ scales as $\ell \sim \xi_\perp \sim \xi_\parallel^\zeta$ with $\zeta = \frac{1}{2}$; and (iii) finally, for $w < \frac{1}{4}$ there is a line of unbinding transitions characterized by w-dependent critical exponents. At these transitions, the longitudinal correlation length diverges as [4]

$$\xi_\parallel \sim (T_* - T)^{-\nu_\parallel} \tag{5}$$

with

$$\begin{aligned} \nu_\parallel &= 1/\sqrt{\tfrac{1}{4}-w} \qquad \text{for } -\tfrac{3}{4} \leqslant w \leqslant \tfrac{1}{4} , \\ &= 1 \qquad\qquad\quad \text{for } w < -\tfrac{3}{4} . \end{aligned} \tag{6}$$

The singular part, f_1, of the interfacial free energy per unit length behaves as $f_1 \sim 1/\xi_\parallel \sim (T_* - T)^{\nu_\parallel}$ to leading order. Therefore, the interfacial energy, $\partial f_1/\partial T$, is continuous for $-\frac{3}{4} \leqslant w \leqslant \frac{1}{4}$ but discontinuous for $w < -\frac{3}{4}$. The transfer-matrix method can also be used to determine the correction terms to the free energy. For $w < -\frac{3}{4}$, one obtains a singular contribution, Δf_1, which scales as

$$\Delta f_1 \sim (T_* - T)^\mu \qquad \text{with } \mu = \sqrt{\tfrac{1}{4}-w} . \tag{7}$$

The different character of the unbinding transition for $w < -\frac{3}{4}$ and $w > -\frac{3}{4}$ can be seen more clearly if one considers the moments, $\langle \ell^n \rangle$. One then finds that

$$\langle \ell^n \rangle \sim \xi_\parallel^{\zeta n} \bar{P}(\xi_\parallel^\zeta) \qquad \text{with } \zeta = \tfrac{1}{2} , \tag{8}$$

where $\bar{P}(\xi_\parallel^\zeta)$ is the probability that the interfacial separation ℓ is of the order of $\xi_\parallel^\zeta$. The relation (8) holds for all $w < \frac{1}{4}$. In the limit of large $\xi_\parallel$, the probability $\bar{P}(\xi_\parallel^\zeta)$ scales as [10]

$$\begin{aligned} \bar{P}(\xi_\parallel^\zeta) &\sim \text{const.} \qquad \text{for } -\tfrac{3}{4} < w \leqslant \tfrac{1}{4} , \\ &\sim \xi_\parallel^{1-\mu} \qquad \text{for } w < -\tfrac{3}{4} , \end{aligned} \tag{9}$$

with μ as in (7). Therefore, interfacial humps with wavelength $\xi_\parallel$ have a transverse extension $\xi_\perp \sim \xi_\parallel^\zeta$ with $\zeta = \frac{1}{2}$ for all values of w. However, the behavior of $\bar{P}(\xi_\parallel^\zeta)$ as given by (9) implies that these humps represent *typical* fluctuations for $-\frac{3}{4} < w \leqslant \frac{1}{4}$ but *exceptional* fluctuations for $w < -\frac{3}{4}$. In the latter case, two $\xi_\parallel$-humps as in fig. 1 are separated by a weakly fluctuating segment of linear size $\sim \xi_\parallel^\mu$.

The case $w = 0$ corresponds to all interactions which decay faster than $1/z^2$ for large

z. This is a rather large class of interaction potentials. The corresponding critical behavior is universal with $\nu_{\parallel}=2$, and $\langle \ell^n \rangle \sim \xi_{\parallel}^{\zeta n}$ as follows from (6), (8), and (9) for $w=0$

The critical behavior just described has been obtained from an explicit solution of the Schrödinger-type equation for specific interaction potentials $V(\ell)$. In addition, a discretized version of the model (3) has been studied by a decimation-type renormalization group (RG) which represents an exact functional renormalization of the interaction $V(\ell)$ [12,5,6].

This exact RG acts in the enlarged function space of interactions, $V(\ell, \ell')$, with the initial value $V_0(\ell, \ell') = \frac{1}{2}[V(\ell) + V(\ell')]$. It exhibits a line of fixed points, $V^*(\ell, \ell')$, which depends only on the product $\ell\ell'$: $V^*(\ell, \ell') = \bar{V}(\ell\ell')$ [5,6]. For small $\ell\ell'$, the fixed points exhibit a logarithmic singularity. In terms of the rescaled potential $U(y) \equiv aV[(Ta/K)^{1/2}y]/T$ which involves the lattice parameter a, one has $U^*(y, y') \approx -(\sigma/2)\ln(yy')$ for small yy'. The strength σ of this singularity, which satisfies $-1 < \sigma < \infty$, can be used to parametrize the line of fixed points. The tails of the fixed points behave as $U^*(y, y') \approx -w/2yy'$ for large yy' with $w = \sigma(2-\sigma)/4$ (Note that $U(y) \approx -w/2y^2$ for large y implies $v(z) \approx -w/z^2$ for large z for any choice of a and ℓ_0).

Since $w = \sigma(2-\sigma)/4$, the fixed point line has a parabolic character and exhibits two branches as first predicted from an approximate functional RG scheme [13]. One branch with $\sigma > 1$ and $-\infty < w < \frac{1}{4}$ corresponds to unbound states of the interfaces. The other branch with $-1 < \sigma < 1$ governs the unbinding transitions and thus leads to the critical behavior as given by (5) and (6) for $-\frac{3}{4} < w \leqslant \frac{1}{4}$. The latter branch ends at $\sigma = -1$ and $w = \frac{3}{4}$. For $\sigma \leqslant -1$, one finds a separatrix for the RG flow which represents an analytic continuation of the fixed-point line. The limiting fixed point at $\sigma = -1$ is identical, for $yy' > 0$, with the fixed point at $\sigma = 3$. The latter fixed point belongs to the other branch of unbound states. In this way, the line of fixed points contains a *closed loop* in function space [5,6].

3. The necklace model for p interfaces in $d = 1+1$

The behavior of several interfaces or domain walls in $d = 1+1$ interacting with a contact potential can be studied in the framework of the necklace model [7-9]. In this model, one considers all configurations of p interfaces which consist of a string of alternating segments, say A and B: In the A segments, all p interfaces are strongly bound together; in the B segments, all p interfaces are unbound from one another.

Fisher and Gelfand [9] have recently studied the necklace model for three dissimilar interfaces with different stiffness constants. They found that the critical behavior at the unbinding transition is nonuniversal depending on these constants and that this critical behavior is analogous to the behavior found for two interfaces as discussed in

section 2 above.

Here, I want to point out that such a correspondence also applies to the necklace model with p similar interfaces with identical stiffness constants. In the latter case, the necklace model has the following features [7,8]: (i) there is no transition for $p<\sqrt{3}$; (ii) for $\sqrt{3}<p<\sqrt{5}$, the longitudinal correlation length diverges with the critical exponent $\nu_{\parallel}=2/(p^2-3)$; and (iii) for $p>\sqrt{5}$, the interfacial energy is discontinuous but the free energy exhibits a singular correction term, $\Delta f_{\rm I}\sim (T_*-T)^{(p^2-3)/2}$. This is identical with the critical behavior as given by (5)-(7) provided w is replaced by

$$w_{\rm eff}=[1-(p^2-3)^2]/4\,. \tag{10}$$

Thus, the unbinding transition for the necklace model with $p=(\sqrt{1-4w}+3)^{1/2}$ similar interfaces has the same critical singularities as the unbinding transition of two interfaces with an interaction, $-w/z^2$.

4. Two strings in $d=1+d_\perp$

Now, let us consider two interacting strings governed by a finite tension in $d=1+d_\perp$. These strings have tangent vectors which point, on average, into the x-direction. Their relative displacement is described by the $d_\perp$-dimensional vector $\boldsymbol{\ell}=(\ell,\ldots,\ell_{d_\perp})$. The configurations of this displacement vector are governed by the effective Hamiltonian

$$\mathcal{H}\{\boldsymbol{\ell}\}=\int \mathrm{d}x\left[\tfrac{1}{2}K\left(\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}x}\right)^2+V(|\boldsymbol{\ell}|)\right], \tag{11}$$

where $V(|\boldsymbol{\ell}|)$ describes the mutual interaction of the strings. For $d_\perp=1$, the model as given by (11) reduces to the Hamiltonian in (1).

It is again convenient to use the rescaled potential $v(z)\equiv 2K\ell_0^2V(\ell_0 z)/T^2$ with $z\equiv|\boldsymbol{\ell}|/\ell_0$. The transfer matrix method then leads to a Schrödinger-type equation with the potential [10]

$$v_{\rm eff}(z)=v(z)+A(d_\perp)/z^2 \tag{12}$$

with

$$\begin{aligned} A(d_\perp)&=\tfrac{1}{4}(d_\perp-1)(d_\perp+1) \qquad \text{for } -1\leqslant d_\perp<1\,,\\ &=\tfrac{1}{4}(d_\perp-3)(d_\perp-1) \qquad \text{for } 1\leqslant d_\perp\,. \end{aligned} \tag{13}$$

The additional term $A(d_\perp)/z^2$ arises from the angular fluctuations of the displacement field. For large $d_\perp$, the amplitude $A(d_\perp)$ is increasing with $d_\perp$ which reflects the fact that the two strings have more and more space into which they can escape.

If the reduced interaction $v(z)$ between the strings behaves as $v(z)\approx w/z^2$ for large z, the critical behavior at the unbinding transition is again described by (4)-(9) with

w replaced by

$$w_{\text{eff}} = w - A(d_\perp) \,. \tag{14}$$

For $d_\perp = 1$ and for $d_\perp = 3$, one has $A(d_\perp) = 0$ from (13). Thus, the critical behavior is identical for these two values of $d_\perp$ with $w_{\text{eff}} = w$. The case $w_{\text{eff}} = w = 0$ then corresponds to the large universality class of interactions which decay faster than $1/z^2$ for large z.

For general $d_\perp$ (i.e., for $d_\perp \neq 1$ and $d_\perp \neq 3$), the large class of string interactions with $w = 0$ is characterized by $w_{\text{eff}} = -A(d_\perp)$. In this case, one obtains from (14) and (7) that

$$\mu = \sqrt{\tfrac{1}{4} + A(d_\perp)} = (d_\perp - 2)/2 \qquad \text{for } d_\perp \geqslant 2 \,. \tag{15}$$

Since $w = -\frac{3}{4}$ now corresponds to $d_\perp = 4$, the probability $\bar{P}(\xi_\parallel^\zeta)$ as given by (9) behaves as

$$\bar{P}(\xi_\parallel^\zeta) \sim \xi_\parallel^{1-\mu} \sim \xi_\parallel^{(4-d_\perp)/2} \qquad \text{for } d_\perp > 4 \,, \tag{16}$$

which applies to all string interactions which decay faster than $1/z^2$ for large z. For large $d_\perp$ or small $\epsilon \equiv 1/d_\perp$, the probability as given by (16) has an essential singularity which cannot be obtained from a perturbative expansion in powers of ϵ.

It would be instructive to obtain the above critical behavior from the exact functional RG discussed in section 2. As a result, one must find a shift of the line of fixed points which reflects the shift of w by $A(d_\perp)$ in (14). In addition, one has to recover the parametrization $w = \sigma(2-\sigma)/4$ for this fixed-point line in $d = 1 + 1$, where σ governs the behavior of the fixed points for small separations, see section 2 above. The simplest parametrization which is consistent with these requirements is given by $w = \frac{1}{4}\sigma(2-\sigma) + A(d_\perp)$.

5. Summary and outlook

In summary, the critical behavior of two interacting manifolds in $d = 1 + d_\perp$ is well understood. First, consider a fixed value of $d_\perp$. Then all interaction potentials $v(z) \sim V(\ell)$, which decay as $v(z) \approx -w/z^2$ for large z, form a separate universality class for each value of w (provided $w_{\text{eff}} = w - A(d_\perp) \leqslant \frac{1}{4}$). In the framework of the RG, these universality classes correspond to a line of nontrivial RG fixed points. Next, let us consider two values of $d_\perp$, say $d_{\perp a}$ and $d_{\perp b}$, and the corresponding universality classes parametrized by w_a and w_b. One then finds from (14) that there is a one-to-one relation between the two sets of universality classes which is given by $w_b = w_a - A(d_{\perp a}) + A(d_{\perp b})$. In this sense, the critical behavior is governed by the same line of fixed points for any value of $d_\perp$.

Likewise, the necklace model for p (similar) interfaces, which interact with a con-

tact interaction, exhibits the same critical behavior as two interfaces interacting with $v(z) \approx -w_{\mathrm{eff}}/z^2$ with w_{eff} as given by (10). This value of w_{eff} should be shifted if one includes power law interactions $\sim -w/|z_i - z_j|^2$ between two neighboring interfaces but this has not been studied so far.

The above systems can be generalized in various ways. In $d = 1 + d_\perp$, one may consider manifolds which are governed by a roughness exponent $\zeta \neq \frac{1}{2}$. Examples are provided, for thermally excited shape fluctuations, by interfaces in quasi-periodic systems or by semi-flexible polymers. Furthermore, one may study manifolds in $d = d_\parallel + d_\perp$ with $d_\parallel \neq 1$. Important examples are two-dimensional interfaces and membranes.

For the case of two manifolds in $d = d_\parallel + 1$, approximate functional RG transformations have been applied [3] which again lead to lines of RG fixed points [13,14]. In fact, one finds a different fixed-point line for each value of $\tau = d_\parallel/\zeta$. In view of the results for $d_\parallel = 1$ and $\zeta = \frac{1}{2}$, it is tempting to speculate that, for fixed τ the same fixed-point line will govern the critical behavior for two or more manifolds and arbitrary $d_\perp$, but this remains to be shown.

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References

[1] For a recent review, see M.E. Fisher, in: Statistical Mechanics of Membranes and Surfaces, eds. D. Nelson, T. Piran and S. Weinberg (World Scientific, Singapore, 1989) p. 19.
[2] For a recent review, see R. Lipowsky, Nature 349 (1991) 475.
[3] R. Lipowsky and M.E. Fisher, Phys. Rev. B 36 (1987) 2126.
[4] R. Lipowsky and T.M. Nieuwenhuizen, J. Phys. A 21 (1988) L89.
[5] F. Jülicher, R. Lipowsky and H. Müller-Krumbhaar, Europhys. Lett. 11 (1990) 657.
[6] H. Spohn, Europhys. Lett. 14 (1991) 689.
[7] M.E. Fisher, J. Stat. Phys. 34 (1984) 667.
[8] D.A. Huse and M.E. Fisher, Phys. Rev. B 29 (1984) 239.
[9] M.E. Fisher and M. Gelfand, J. Stat. Phys. 53 (1988) 175.
[10] R. Lipowsky, Europhys. Lett., in press.
[11] D.M. Kroll and R. Lipowsky, Phys. Rev. B 28 (1983) 5273.
[12] D.A. Huse, Phys. Rev. Lett. 58 (1987) 176.
[13] R. Lipowsky, Phys. Rev. Lett. 62 (1989) 704; Physica Scripta T29 (1989) 259.
[14] S. Grotehans and R. Lipowsky, Phys. Rev. A 41 (1990) 4574.