Delocalization transitions of low-dimensional manifolds

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(Received 23 August 1991)

The behavior of low-dimensional manifolds such as interfaces, membranes, or polymers in an external potential is studied in $d=d_{\parallel}+d_{\perp}$ dimensions. If the potential has several minima, the manifolds can undergo nontrivial transitions from a localized to a delocalized state at a characteristic temperature $T=T_*$. For interfaces in $d=1+1$, several universality classes must be distinguished which can be determined exactly by transfer-matrix methods. The same classification also applies to (self-intersecting) polymers in $d=1+d_{\perp}$. Within a functional renormalization-group approach, the critical behavior is governed by a line of fixed points in close analogy with unbinding transitions. However, in contrast to unbinding, this fixed point line now contains the trivial fixed point at which one can determine the spectrum of eigenperturbations exactly. The number of relevant eigenperturbations increases monotonically with the decay exponent $\tau=d_{\parallel}/\zeta$ where the roughness exponent ζ characterizes the delocalized manifold.

PACS number(s): 68.45.−v, 64.60.Fr, 82.70.−y

I. INTRODUCTION

Consider a low-dimensional manifold subject to an external potential. Examples are (i) a two-dimensional interface between two bulk phases in a homogeneous gravitational field, (ii) an interface or domain wall in two or three dimensions between two ordered domains in the presence of a linear or a planar defect, and (iii) a polymer interacting with a fluid layer or membrane. In all cases, the external potential V depends on a single coordinate z and is symmetric, i.e., $V(-z)=V(z)$. In addition, $V(z)$ has an attractive part at small z which acts to *localize* the manifold.

Now, assume that we change a control parameter such as temperature or the potential strength, $V(0)$. Then, the manifold may undergo a *delocalization transition* from a localized to a delocalized state where the manifold is rough and makes arbitrarily large excursions from its planar reference state.

In the following, we will focus on $(d-1)$-dimensional interfaces and membranes in d dimensions. In these cases, the configuration of the manifold can be described by the scalar displacement field, $z=z(\mathbf{x})$ which depends on the $(d-1)$-dimensional coordinate $\mathbf{x}$. Our results for interfaces or domain walls in $d=1+1$ also apply to self-intersecting or Gaussian polymers in $d=1+d_{\perp}$.

At a delocalization transition, the roughness $\xi_{\perp}$ of the manifold as measured by the second cumulant of its displacement field z via

$$\xi_{\perp}^2=\langle[z-\langle z\rangle]^2\rangle \tag{1.1}$$

goes to infinity. In fact, we will focus on the situation where the mean position $\langle z\rangle$ for the localized state satisfies $\langle z\rangle=0$. The latter property is always guaranteed if the symmetric potential has its global minimum at $z=0$. In addition, the relation $\langle z\rangle=0$ for the localized state also holds for *all* symmetric potentials in $d=1+1$. Indeed, consider a potential $V(z)$ which exhibits two local minima at $z=\pm z_0$ with $z_0<\infty$. In this case, the displacement field "tunnels" through any intermediate potential barrier (of finite height), see Sec. III D. On the other hand, the potential may have local minima at $z=\pm\infty$. In the latter case, a spontaneous symmetry breaking can only occur for the delocalized state.

The roughness $\xi_{\perp}$ also diverges at *unbinding transitions* [1] of two interacting manifolds. Examples are wetting transitions [2,3], adhesion transitions of membranes [4], and adsorption transitions of polymers [5]. In these cases, the interaction of the manifolds is described by an effective potential $V(z)$ which contains a hard wall, i.e., $V(z)=\infty$ [or $V(z)$ is equal to a large constant] for $z<0$ because the two manifolds cannot penetrate each other. This implies that *both* $\langle z\rangle$ *and* $\xi_{\perp}$ as given by (1.1) diverge at an unbinding transition.

The critical behavior at unbinding can be studied by functional renormalization. Using two different approximate renormalization-group (RG) schemes, we have recently found a complex fixed-point structure [6]. There is a line of RG fixed points for each value of $\tau=d_{\parallel}/\zeta$ where $d_{\parallel}$ is the spatial dimensionality of the manifolds and ζ is the roughness exponent of the unbound (or delocalized) manifolds. As τ is increased, this line of fixed points undergoes a sequence of bifurcations. In the present paper, we show that, within the framework of functional renormalization the critical behavior at delocalization is also governed by a line of RG fixed points. In addition, this fixed point line again undergoes a sequence of bifurcations. There is, however, one crucial difference between the two fixed point lines for unbinding and delocalization. As shown below, the fixed point line for delocalization contains the trivial fixed point, $V^*(z)\equiv 0$, at which one can determine the *exact* eigenperturbations. This provides additional evidence that the sequence of bifurcations is an intrinsic property of the models for delocalization (or unbinding) and, thus, is not an artifact of the ap-

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proximations involved in the RG.

One open problem for wetting transitions is the critical behavior in the marginal dimension [7–10]. For systems without quenched or frozen randomness, this marginal dimension is $d=3$. In the framework of the RG, this uncertainty arises from the fact that there is no analytical limit of the fixed-point potentials as one approaches the marginal dimension from below [6,11,12]. In contrast, it will be shown below, that an analytical limit exists for delocalization phenomena even though the same *bifurcation* structure of fixed points arises as for unbinding or wetting.

This paper is organized as follows: In Sec. II we describe the models for delocalization of interfaces, membranes, and (Gaussian) polymers. Although there are fundamental differences between these manifolds, they can be treated in the same framework of functional renormalization.

In Sec. III, we describe delocalization of interfaces in $d=1+1$ dimensions for different potentials using exact transfer-matrix methods. In this way, we obtain *all* universality classes or scaling regimes for interfaces in $d=1+1$. The same classification also applies to Gaussian or self-intersecting polymers in $d=1+d_\perp$. These results serve as a starting point for linear RG calculations, which are described in Sec. IV and which extend the transfer-matrix calculations to manifolds in general $d=d_\parallel+1$ with arbitrary roughness. The nonlinear RG flow is investigated in Sec. V. Here we get an overview over the global RG flow, and the above-mentioned bifurcation structure is explained. Finally, we consider membranes under tension or interfaces in three dimensions in Sec. VI.

II. MODELS FOR DELOCALIZATION PHENOMENA

Consider an interface or membrane with a local distance $z(\mathbf{x})$ from a reference plane, where $\mathbf{x}$ is a $(d-1)$-dimensional coordinate parallel to this plane. The undulations of the surfaces are governed by the free energy per unit area which is given by $\frac{1}{2}K[\nabla^n z(\mathbf{x})]^2$. For membranes, K is the effective bending rigidity [13,14] and n is related to the roughness exponents ζ via $n=d_\parallel/2+\zeta$ (Ref. [1]). In $d=2+1$ fluid membranes have $\zeta=1$ while polymerized or solidlike membranes have $\zeta=0.5$ [15]. The roughness exponent ζ describes the fluctuations of the free membrane: for lateral distance $L_\parallel$, the typical value $L_\perp$ of the transverse fluctuations will scale as $L_\perp \sim L_\parallel^\zeta$, provided $L_\parallel$ is small compared to the lateral membrane size. In the case of interfaces, K is the effective interfacial stiffness and $n=1$. For a $(d-1)$-dimensional interface, the roughness exponent ζ is given by $\zeta=(3-d)/2$.

The membrane or interface configurations are then governed by the effective Hamiltonian [1]

$$\mathcal{H}\{z\}=\int d^{d-1}x\left(\tfrac{1}{2}K[\nabla^n z(\mathbf{x})]^2+V(z)\right) . \tag{2.1}$$

The second term, $V(z)$, represents a symmetrical external potential. The Hamiltonian $\mathcal{H}(z)$ implicitly contains a short distance cutoff, a, of the order of the molecular size.

Self-intersecting polymers can be described by the effective Hamiltonian [16]

$$\mathcal{H}\{\mathbf{r}\}=\int_0^{Na} ds\, a^{-1}\left[\tfrac{1}{2}T\left[\frac{d\mathbf{x}}{ds}\right]^2+\tfrac{1}{2}T\left[\frac{dz}{ds}\right]^2+V(z)\right] . \tag{2.2}$$

Here, $V(z)$ is the potential energy of one monomer at distance z from the origin. The configuration of the polymer is described by $\mathbf{r}(s)=(\mathbf{x}(s),z(s))$ where s labels the sequence of N monomers. Since the lateral coordinates $\mathbf{x}$ and z decouple the problem can be solved exactly and is equivalent to delocalization of an interface in $d=1+1$.

In this paper we present (i) exact calculations for interfaces in $d=1+1$ and (ii) approximate renormalization-group calculations in arbitrary dimensions. The approximate RG can be applied both to interfaces and to membranes: the corresponding RG transformation is identical apart from one parameter, the so-called decay exponent, $\tau=(d-1)/\zeta$. We therefore study the RG as a function of this parameter τ. The exact calculations for $d=1+1$ and $\tau=2$ will be useful in order to check the validity of the RG.

III. DELOCALIZATION OF INTERFACES IN TWO DIMENSIONS

In this section, we consider the interfacial behavior in a two-dimensional system which contains a linear defect. A simple example is provided by a two-dimensional Ising model with nearest-neighbor couplings J_b and a line of weakened bonds, $J_0<J_b$ [17–19]. Then, an interface which runs, on average, parallel to this linear defect feels an effective square-well potential (since the interface is always rough for $T>0$, the periodic lattice potential is irrelevant). One may also consider the situation in which the linear defect at $z=0$ leads to a long-ranged perturbation of the coupling constants, and $J=J(z)$ with $J(z)\approx J_b-C/|z|^r$ for large z. In this case, the effective interface potential, $V(z)\simeq J(z)/a$, also has a long-ranged part.

A. Transfer-matrix formalism

For $d=1+1$, the interface variable z depends only on one spatial coordinate and its statistical properties can be evaluated via transfer-matrix methods [20–22]. For $n=1$, the transfer matrix becomes a Hamilton operator in the limit of zero small-scale cutoff, $a\to 0$. One then has to determine the eigenvalues E_n and eigenfunctions ϕ_n of this Hamilton operator which satisfy

$$\hat{H}\phi_n=\left[-\frac{T^2}{2K}\frac{\partial^2}{\partial z^2}+V(z)\right]\phi_n=E_n\phi_n , \tag{3.1}$$

where the interface potential $V(z)$ now plays the role of a "quantum-mechanical" potential.

The physical quantities of interest can be expressed in terms of eigenvalues and eigenfunctions of this Schrödinger-type equation. The interfacial energy is sim-

ply given by

$$\Sigma_I = E_0 = (\phi_0|\hat{H}\phi_0) , \tag{3.2}$$

where E_0 is the ground-state energy, and ϕ_0 is the normalized ground-state wave function. Here and below, the parentheses indicate a scalar product: $(a|b)=\int_{-\infty}^{\infty} dz\, a^*(z)b(z)$. The average position $l=\langle z\rangle = (\phi_0|z\phi_0)$ vanishes since the potential is taken to be symmetric. The roughness $\xi_\perp$ is given by

$$\xi_\perp^2 = \langle (z-l)^2\rangle = (\phi_0|(z-l)^2\phi_0) . \tag{3.3}$$

Finally, the correlation length $\xi_\parallel$ follows from the expression for the difference correlation function and is given by

$$\xi_\parallel = \frac{T}{E_1 - E_0} . \tag{3.4}$$

B. Square-well potential

First, we consider the simple case of a square-well potential given by

$$V = \begin{cases} G & \text{for } -z_0 < z < z_0 \\ 0 & \text{elsewhere} . \end{cases} \tag{3.5}$$

As long as $G<0$, this potential has a localized ground state ϕ_0 with $E_0<0$ which is easily calculated. It then follows that the interface is always localized as long as $G<0$ [19,23]. As G approaches zero from below, the ground-state energy vanishes as $E_0 \approx -2z_0^2 K G^2/T^2$, and the interface delocalizes. Furthermore, the first excited state has energy $E_1=0$ for small $|G|$. This implies the critical behavior

$$\xi_\perp \approx (T^2/K\sqrt{8}z_0)|G|^{-\nu_\perp} \quad \text{with } \nu_\perp = 1 , \tag{3.6}$$

$$\xi_\parallel \approx (T^3/K2z_0^2)|G|^{-\nu_\parallel} \quad \text{with } \nu_\parallel = 2 , \tag{3.7}$$

and

$$\Sigma_I \approx -(2z_0^2 K/T^2)|G|^{2-\alpha} \quad \text{with } \alpha = 0 . \tag{3.8}$$

The two length scales $\xi_\perp$ and $\xi_\parallel$ satisfy the relation $\xi_\perp \approx (\mathcal{D}_\infty T/K)^{1/2}\xi_\parallel^\zeta$ with the roughness exponent $\zeta=\frac{1}{2}$ and $\mathcal{D}_\infty = \frac{1}{4}$.

C. Potentials with power-law tails

Next we consider the case of long-ranged potentials V which behave as $V \approx G|z|^{-r}$ for large values of $|z|$ with $Gr<0$ and with only one minimum at $z=0$. Since these potentials are even more attractive than the square-well potential, the interface is again localized as long as $|G|>0$ and delocalizes only as $|G|$ tends to zero. For small G, one can employ exact results for the associated Hamilton operator [24] to determine the range of r where the exponents as given by (3.6)–(3.8) still apply. These results show that one has at least one bound state if the integral $\int dx\, V(x) \le 0$. In particular, for $r>2$, one has a bound state only for $\int dx\, V(x)<0$, and, for small G, the phase boundary is implicitly given by

$$\int dx\, V(x) = 0 . \tag{3.9}$$

In the case $1<r\le 2$, one has infinitely many bound states for any $G<0$ and a lower bound to the phase boundary is given by (3.9). For $r=1$, the integral $\int dx\, V(x)$ diverges indicating that the long-ranged power-law tail of V becomes important for the critical behavior. So we conclude that the critical exponents in (3.6)–(3.8) are valid provided $r>1$. This can be understood from the renormalization-group transformation discussed in Sec. IV.

D. Potentials with $1/z^2$ tails

Next, let us investigate the critical behavior for the potentials defined by

$$V(z) = \begin{cases} U & \text{for } -z_0 < z < z_0 \\ W/z^2 & \text{for } |z| \ge z_0 . \end{cases} \tag{3.10}$$

where U and W can be positive *or* negative. These potentials can have three extrema, and a maximum rather than a minimum at the origin.

The detailed calculation for this type of potential is described in Appendix A. Here we quote only the results for the delocalization process. It is convenient to use the rescaled variables $u \equiv 2Kz_0^2 U/T^2$, $w \equiv 2KW/T^2$. As shown in Appendix A, the phase boundary $u_c = u_c(w)$ is given by

$$\sqrt{u_c}\tanh\sqrt{u_c} - \tfrac{1}{2} = -\mu \quad \text{for } u>0 \tag{3.11a}$$

and by

$$-\sqrt{|u_c|}\tan\sqrt{|u_c|} - \tfrac{1}{2} = -\mu \quad \text{for } u<0 \tag{3.11b}$$

with $\mu = (\frac{1}{4}+w)^{1/2}$. Note that for $w=0$, $u_c=0$ is a solution of (3.11a) and (3.11b). For small values of w, an expansion of (3.11a) and (3.11b) around $w=0$ and $u_c=0$ leads to

$$u_c \approx w \quad \text{for small } w . \tag{3.12}$$

The transition line for delocalization is shown in Fig. 1. Note that the tangent $u_c=w$ at $w=0$ is consistent with the lower bound to the phase boundary as given by (3.9) in the limit of small u_c and w. As w approaches $-\frac{1}{4}$ from above, the transition line $u_c(w)$ approaches the limit $u_c(-\frac{1}{4}) = u_{mc} = 0.595$. For large w, the transition line approaches the asymptotic value $u_c(\infty) = -\pi^2/4$.

These results show that delocalization transitions are governed by the same interplay of the attractive and repulsive part of the potential as wetting transitions [22]. An attractive short-range part of the potential cannot localize the interface if the repulsive part of the potential is strong enough.

The same asymptotic analysis as in the two-dimensional wetting case gives the ground-state energy. The behavior of the rescaled ground-state energy $e_0 = 2Kz_0^2 E_0/T^2 \equiv -q^2$ in the vicinity of the transition line can be calculated by using the expansion

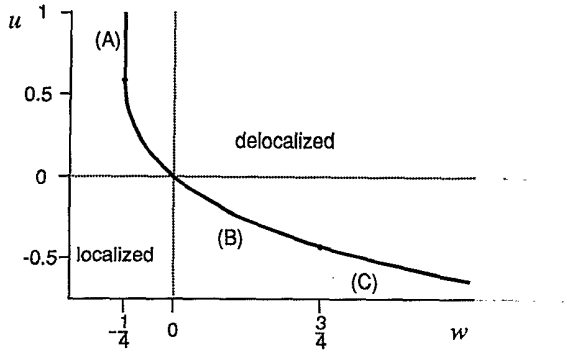

FIG. 1. Phase diagram for delocalization in $d=1+1$ from a potential as defined in (3.10). The locus of transition consists of three parts: (A) For $u>u_{mc}=0.595$, the transition line is given by $w=-\frac{1}{4}$; for $u<u_{mc}$ with (B) for $-\frac{1}{4}<w<\frac{3}{4}$, and ($C$) for $w>\frac{3}{4}$, the transition line follows from (3.11) and (3.12).

$u=u(w_c)+\delta u$. One then has to repeat the analysis as presented in Appendix A for the next order in δu and q. As a result, one obtains that e_0 has the asymptotic behavior given by

$$|e_0|\sim\begin{cases}\exp(-c/\delta u) & \text{for } \mu=0 \quad (3.13)\\ |\delta u|^{1/\mu} & \text{for } 0<\mu<1 \quad (3.14)\\ |\delta u|/\ln|\delta u| & \text{for } \mu=1 \quad (3.15)\\ |\delta u| & \text{for } \mu>1\ , \quad (3.16)\end{cases}$$

with $\mu=(\frac{1}{4}+w)^{1/2}$ as before. For $\mu>1$ one has a regime with unusual first-order transitions which occurs also for unbinding or wetting transitions (the so-called regime C).

The asymptotic behavior of e_0 for $w<-\frac{1}{4}$ can be determined as in Ref. [22] by rewriting the matching condition (A7) in Appendix A. One then finds that there is still a bound state for $u<u_c$ as w crosses the line at $w_c^-=-\frac{1}{4}$. For $u>u_c$, on the other hand, the ground-state energy, $e_0=-q^2$, goes to zero as w approaches $w_c^-=-\frac{1}{4}$ from below. Therefore the phase boundary is given by $w_c(u)=-\frac{1}{4}$ for $u>u_{mc}$; see Fig. 1.

In the regimes B and C the energy of the first excited state e_1 is zero close to the transition line. It then follows from (3.14)–(3.16) that the correlation length $\xi_\parallel=T/(E_1-E_0)=-T/E_0$ behaves as

$$\xi_\parallel\sim|\delta u|^{-\nu_\parallel} \quad \text{for } w>-\tfrac{1}{4} \quad (3.17)$$

with

$$\nu_\parallel=\begin{cases}1/(\frac{1}{4}+w)^{1/2} & \text{for } -\frac{1}{4}<w<\frac{3}{4}\\ 1 & \text{for } \frac{3}{4}<w\ .\end{cases} \quad (3.18)$$

IV. LINEAR RENORMALIZATION OF SYMMETRIC POTENTIALS

A. Linear recursion relations

The model defined by (2.1) will now be studied by functional RG methods. In general, such RG methods act as nonlinear maps within the function space of potentials, $V(z)$ [11,25–30]. The nonlinear part of these RG maps is usually approximate and depends, to some extent, on the form of the cutoff procedure. Their linear part, on the other hand, is unique and consists of two relatively simple transformations: (i) The (reduced) stiffness K/T which parametrizes the effective Hamiltonian $\mathcal{H}_0\{z\}$ is *not* changed at all; and (ii) the potential is transformed according to the recursion relation

$$V^{(N+1)}(z)=\mathcal{L}[V^N(z)]\ , \quad (4.1)$$

where $\mathcal{L}$ is the linear operator defined by

$$\mathcal{L}[V(z)]\equiv b^{d-1}\exp\left[\frac{1}{2}\left[\bar{a}_\perp^2\frac{\partial^2}{\partial y^2}\right]\right]V(y)\Big|_{y=b^\zeta z}\ . \quad (4.2)$$

This differential operator can be rewritten as the integral operator

$$\mathcal{L}[V(z)]=b^{d-1}\int_{-\infty}^{\infty}dz'\frac{1}{\sqrt{2\pi}\bar{a}_\perp}\exp[-\tfrac{1}{2}(b^\zeta z-z')^2/\bar{a}_\perp^2]\times V(z')\ . \quad (4.3)$$

In these expressions, $b>1$ is the usual rescaling factor, and $\bar{a}_\perp$ represents the roughness of the small-scale fluctuations, $z_>$, which is given by

$$\bar{a}_\perp^2\equiv\langle z_>^2\rangle=\frac{T}{K}\left[\int_p^\Lambda-\int_p^{\Lambda/b}\right]p^{-2n}\ , \quad (4.4)$$

where $\int_p^\Lambda$ denotes a momentum integral which is truncated at large momenta by the cutoff $\Lambda\simeq 1/a$.

B. Matching procedure

For potentials which keep the manifold localized until the potential strength $|G|$ is decreased to zero, the delocalization transition is governed by the trivial fixed point $V^*(z)\equiv 0$ and thus by the effective Hamiltonian $\mathcal{H}_0\{z\}$. By construction, this Hamiltonian already corresponds to a fixed-point Hamiltonian $\mathcal{H}^*\{z\}$ which will be referred to as a Gaussian fixed point. In fact, there is a whole line of such fixed points which is parametrized by the reduced stiffness K/T. Thus the *linear* RG transformation as given by (4.2) or (4.3) represents the linearization of the nonlinear RG around these Gaussian fixed points. When viewed in the function space of potentials, these Gaussian fixed points are characterized by the trivial fixed point $V^*(z)=0$, and the linear RG gives the RG flow close to this fixed-point potential. During one RG step with rescaling factor b, the longitudinal correlation length $\xi_\parallel(b)$ is reduced by a factor of b as a result of the scale transformation: $\xi_\parallel(b)=\xi_\parallel/b$. Several, subsequent applications of the linear RG can be replaced by one RG step with a larger value of b since the linear recursion relation (4.3)

has the semigroup property. Now, choose $b=b_0$ in such a way that the new correlation length becomes $\xi_\parallel(b_0)\simeq a$. Then, $\xi_\parallel(b_0)$ is sufficiently small that it can be obtained from mean-field (MF) theory which implies

$$\frac{K}{a^{2n}}\simeq\frac{K}{\xi_\parallel^{2n}(b_0)}=\frac{\partial^2\mathcal{L}[V(z)]}{\partial z^2}\bigg|_{z=0} . \tag{4.5}$$

Then, the longitudinal correlation length $\xi_\parallel$ of the initial potential follows from

$$\xi_\parallel\approx b_0\xi_\parallel(b_0)=b_0 a . \tag{4.6}$$

From the definition of $\bar{a}_\perp$ in (4.4), one has

$$\bar{a}_\perp^2\sim b_0^{2\zeta} \quad \text{for } b_0\gg 1 . \tag{4.7}$$

This leads to

$$\frac{K}{\xi_\parallel^{2n}(b_0)}=b_0^{(d-1)}\frac{\partial^2}{\partial z^2}\int_{-\infty}^{\infty}dz'\frac{1}{\sqrt{2\pi}\bar{a}_\perp}\exp[-\tfrac{1}{2}(b_0^\zeta z-z')^2/\bar{a}_\perp^2]V(z')|_{z=0} \tag{4.8}$$

$$=b_0^{d-1}b_0^{2\zeta}\bar{a}_\perp^{-2}\int_{-\infty}^{\infty}dz'\frac{1}{\sqrt{2\pi}\bar{a}_\perp}\exp[-\tfrac{1}{2}(z'/\bar{a}_\perp)^2]\left[\frac{z'^2}{\bar{a}_\perp^2}-1\right]V(z') . \tag{4.9}$$

C. Short-ranged Gaussian potential

The matching procedure as described above will be applied to examine the critical behavior for different types of potentials. First consider a short-ranged Gaussian potential:

$$V(z)=G\exp\left[-\frac{1}{2}\left[\frac{z}{z_0}\right]^2\right] , \tag{4.10}$$

with $G<0$. From the transfer-matrix calculations in two dimensions, one knows that the interface is localized as long as $G<0$. From this we conclude that for any dimension $2\le d\le 3$ the matching procedure can be employed since fluctuations are expected to become less important with increasing dimensionality.

Inserting this potential into (4.9), we obtain

$$\frac{K}{\xi_\parallel^{2n}(b_0)}=b_0^{d-1}b_0^{2\zeta}\bar{a}_\perp^{-2}[|G|z_0 a_\perp^{-1}+O(a_\perp^{-3})] . \tag{4.11}$$

If $\bar{a}_\perp$ is expressed in terms of b_0, via Eq. (4.7), the matching condition (4.6) leads to

$$\xi_\parallel\sim|G|^{-\nu_\parallel} \quad \text{with } \nu_\parallel=1/[\zeta(\tau-1)] , \tag{4.12}$$

with $\tau=(d-1)/\zeta$ valid in all dimensions $2\le d\le 3$. Interfaces in the square-well potential have a critical exponent $\nu_\parallel=2$ in $d=1+1$ as in (3.7). If the corresponding values $\tau=2$ and $\zeta=\frac{1}{2}$ are inserted into (4.12), one obtains $\nu_\parallel=2$ which shows that the linear RG gives the correct critical behavior. It is assumed here implicitly that the Gaussian and the square-well potential show the same critical behavior since they are both short ranged. In Sec. IV D the class of potentials which belong to this universality class is determined.

D. Long-ranged potentials

Now potentials of the form

$$V(z)=G(z^2+z_0^2)^{-r/2} \tag{4.13}$$

with $G<0$ and $r>0$ are considered. Since these potentials are even more attractive than the Gaussian potential (and have no repulsive part) the matching procedure can be employed again for $2\le d\le 3$. Asymptotic analysis of (4.9) for the potential as given by (4.13) leads to

$$\frac{K}{\xi_\parallel^{2n}(b_0)}\approx|G|b_0^{d-1+2\zeta}\bar{a}_\perp^{-2}(c_1 z_0^{1-r}\bar{a}_\perp^{-1}+c_r\bar{a}_\perp^{-r}) , \tag{4.14}$$

where c_1 and c_r are dimensionless coefficients. Using the same procedure as before, one obtains now the critical behavior

$$\xi_\parallel\sim|G|^{-\nu_\parallel} \tag{4.15a}$$

with

$$\nu_\parallel=\frac{1}{\zeta(\tau-r)} \quad \text{for } r<1 , \tag{4.15b}$$

$$\nu_\parallel=\frac{1}{\zeta(\tau-1)} \quad \text{for } r>1 . \tag{4.15c}$$

From (4.15) one sees that all potentials with $r>1$ have the same exponent $\nu_\parallel$ as the Gaussian potential and therefore show the same *universal* behavior. The borderline case is $r=1$; for $r<1$, the exponent depends on r.

E. Perturbations around the trivial fixed point

In Sec. IV D we used a matching procedure in order to obtain the critical exponent $\nu_\parallel$. Alternatively, one may study the eigenperturbations around the fixed-point potentials of the RG flow. In the latter context, physical potentials are treated as perturbations and one has to determine the leading eigenvalue of such a perturbation.

First consider a long-ranged potential which again behaves as

$$V(z)\approx G|z|^{-r} \quad \text{for large } |z| , \tag{4.16}$$

with $Gr<0$. Using the differential operator form (4.2) for

the linear RG operator, $\mathcal{L}$, one immediately sees that the power-law tail in (4.16) is mapped onto the power-law tail $G'|z|^{-r}$ with renormalized potential strength

$$G'=b^{\lambda_{LR}}G \quad \text{with } \lambda_{LR}=\zeta(\tau-r) . \tag{4.17}$$

If one uses the same procedure for a short-ranged potential given by

$$V(z)=G\exp\left[-\frac{1}{2}\left[\frac{z}{z_0}\right]^2\right] , \tag{4.18}$$

with $G<0$, one obtains another Gaussian potential with renormalized parameters

$$G'=b^{d-1}G/[(1+(\bar{a}_\perp/z_0)^2]^{1/2} \tag{4.19}$$

and

$$z_0'=\bar{a}_\perp[1+(z_0/\bar{a}_\perp)^2]^{1/2}/b^{\zeta} . \tag{4.20}$$

For $\zeta>0$, the length scale z_0 is mapped onto the fixed value $z_0^*=\bar{a}_\perp/(b^{2\zeta}-1)^{1/2}$ after a few iterations of the RG transformation. When this fixed value is inserted into the relation (4.19), one obtains the simple recursion relation

$$G'=b^{\lambda_{SR}}G \quad \text{with } \lambda_{SR}=\zeta(\tau-1) . \tag{4.21}$$

A physical potential may contain both long-ranged and short-ranged parts. Now, if we compare (4.15) with (4.17) and (4.21), we find that the result of the matching procedure is equivalent to the simple rule $\nu_\parallel=1/\lambda$, provided λ is identified with $\max(\lambda_{LR},\lambda_{SR})$ where $\lambda_{LR}=\zeta(\tau-r)$ and $\lambda_{SR}=\zeta(\tau-1)$.

F. Competition of short-ranged and long-ranged potentials

Now, consider the more general potential defined by

$$V(z)=\begin{cases} U & \text{for } |z|<z_0 \\ W/z^r & \text{for } |z|>z_0 . \end{cases} \tag{4.22}$$

Using the linear RG, one can compute the tangent to the phase boundary in the (W,U) plane around the origin and compare the RG results from matching with the exact results as found in two dimensions; see Sec. III D.

The explicit calculation can be found in Appendix B. As a result one finds

$$\frac{K}{\xi_\parallel^{2n}(b_0)}\sim b_0^{d-1}\left[\bar{a}_\perp^{-1}\left[-2z_0U-\frac{2Wz_0^{1-r}}{r-1}\right]+\bar{a}_\perp^{-3}\left[z_0^3U+\frac{3z_0^{3-r}W}{r-3}\right]+O(\bar{a}_\perp^{-r^<})\right] , \tag{4.23a}$$

for $r>3$ with $r^<=\min(r,5)$ and

$$\frac{K}{\xi_\parallel^{2n}(b_0)}\sim b_0^{d-1}\left[\bar{a}_\perp^{-1}\left[-2z_0U-\frac{2Wz_0^{1-r}}{r-1}\right]+\bar{a}_\perp^{-r}Wc_r^{<}+O(\bar{a}_\perp^{-3})\right] \tag{4.23b}$$

for $1<r<3$ where $c_r^<$ is a dimensionless coefficient.

For $W=0$, i.e., in the absence of a long-ranged part, this implies $\nu_\parallel=1/\zeta(\tau-1)$ as previously obtained for the Gaussian potential; see (4.12). For $W\neq0$, the exponent $\nu_\parallel$ remains unchanged until $r<1$. In the latter case, the long-ranged part of the potential dominates: $\nu_\parallel=1/\zeta(\tau-r)$ and the transition occurs at $W=0$. This agrees with the results derived in the Secs. IV C and IV D.

In addition, for $r>1$, the linear RG predicts that the tangent to the phase boundary at the origin is given by

$$U=-\frac{Wz_0^{-r}}{r-1} , \tag{4.24}$$

where the leading term in (4.23) vanishes. This is consistent with the exact results in $d=2$ because (4.24) implies $\int dz\, V(z)=0$.

In the linear RG approach, the exponent $\nu_\parallel$ is independent of the locus of the transition and thus independent of W. This is a contradiction to the exact result for $d=1+1$ and $\tau=2$ for which $\nu_\parallel$ is nonuniversal; see (3.18). The reason is that the RG can only be trusted at the Gaussian fixed point, i.e., in the vicinity of $(W,U)=(0,0)$. To get the nonuniversal behavior of $\nu_\parallel$ along the transition line, a nonlinear RG transform must be employed. This will be done in Sec. V.

Inspection of (4.23) shows that there is a multicritical point at $W_c=U_c=0$ if (i) $\tau>3$ or (ii) $\tau>r$ since then the $\bar{a}_\perp^{-3}$ or the $\bar{a}_\perp^{-r}$ term becomes relevant. The corresponding correction terms diverge as $b_0^{\zeta(\tau-3)}$ or $b_0^{\zeta(\tau-r)}$, respectively, for large $b_0\sim\xi_\parallel$.

For $\tau=2$ and $1<r<2$ this indicates a weak fluctuation regime with first-order transitions in the region of the phase diagram with $W<0$. If τ is increased beyond $\tau=3$, there will also be first-order transitions and nonuniversal critical behavior as described in Sec. V.

V. NONLINEAR RENORMALIZATION OF SYMMETRIC POTENTIALS

A. Renormalization-group approach for general τ

In the preceding section we have seen that interfaces delocalize at $G^*=0$, provided the potential has only one minimum at $z=0$ and no additional extrema. On the other hand, our exact transfer-matrix calculations in $d=1+1$ have shown that it is possible to have a delocalization at $G^*\neq0$ provided the potential has several extre-

ma. In this section, we generalize the results for $d=1+1$ and $\tau=2$ to arbitrary τ. Since we anticipate nontrivial phase boundaries, we cannot use the linear RG. Instead, a nonlinear approximate RG transformation based on Wilson recursions relations [25] will be used. In the infinitesimal rescaling limit, one then obtains the flow equation [11,30]

$$\frac{\partial U}{\partial s}=\tau U+yU'(y)+\ln(1+U'') \tag{5.1}$$

for the rescaled variables $U\sim V$ and $y\sim z$, where U' denotes $\partial U/\partial y$ and $\tau=(d-1)/\zeta$. We need the fixed points $U^*(y)$ and the eigenperturbations $f_\lambda(y)$ of this RG flow in order to determine the critical behavior. The fixed points satisfy

$$0=\tau U^*+yU^{*\prime}+\ln(1+U^{*\prime\prime})\ , \tag{5.2}$$

and the eigenperturbations follow from

$$0=(\tau-\lambda/\zeta)f_\lambda+yf_\lambda'+\frac{f_\lambda''}{1+U^{*\prime\prime}}\ . \tag{5.3}$$

These equations must be supplemented by appropriate boundary conditions. We are interested in symmetric potentials and choose

$$\Sigma\equiv U^*(0)\ \text{ and }\ U^{*\prime}(0)=0\ . \tag{5.4a}$$

The condition that $U^*(y)$ decays to zero for large $|y|$ implies two independent solutions for $U^*(y)$:

$$U^*(y)\approx\rho_L|y|^{-\tau}+\rho_S|y|^{\tau-1}\exp(-y^2/2)$$
$$\text{for large } |y| \tag{5.4b}$$

which have the amplitudes $\rho_L(\Sigma)$ and $\rho_S(\Sigma)$. Thus one has a *line of fixed points* parametrized by Σ.

The linear eigenperturbations $f_\lambda(y)$ for each fixed-point solution U^* are also taken to be symmetric: $f_\lambda(0)=1$, $f_\lambda'(0)=0$. For large $|y|$, the functions behave as

$$f_\lambda\approx C_f|y|^{-\tau+\lambda/\zeta}+D_f|y|^{\tau-1-\lambda/\zeta}\exp(-y^2/2)\ . \tag{5.5}$$

The amplitudes C_f, D_f depend on Σ and on λ/ζ. As for unbinding phenomena, the condition that the eigenperturbations f_λ should not dominate the fixed point U^* leads to the condition [6,30]

$$C_f(\Sigma,\lambda/\zeta)=0\ . \tag{5.6}$$

This condition selects an infinite number of discrete eigenvalues λ. The line of fixed points and the eigenperturbations will now be obtained numerically.

B. The case $\tau=2$

Figure 2(a) shows the function ρ_L and the most relevant eigenperturbation λ_1/ζ as a function of Σ for $\tau=2$, i.e., for interfaces in $d=2$ dimensions or for fluid membranes in $d=3$. At the minimum of the function $\rho_L(\Sigma)$, the eigenvalue λ_1 goes through zero and the eigenperturbation is marginal. The eigenvalue of this eigenperturbation increases with decreasing Σ. So we have

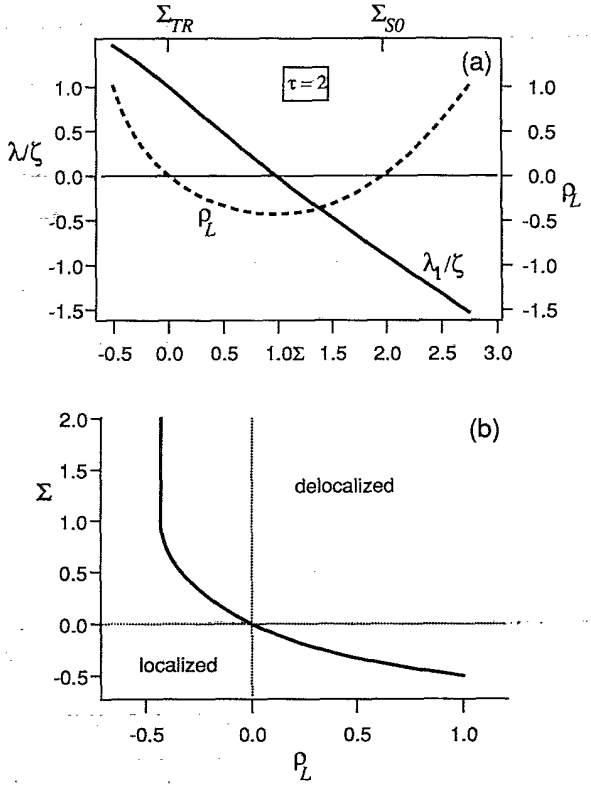

FIG. 2. (a) Line of fixed points parametrized by ρ_L and the largest eigenvalue λ_1/ζ as a function of $\Sigma=U^*(0)$ for $\tau=2$. The function ρ_L has two zeros which corresponds to two short-ranged fixed points: the trivial one and the nontrivial one have one and no relevant eigenperturbation, respectively. (b) Phase diagram for delocalization with $\tau=2$ derived from (a). The amplitude Σ is plotted vs the branch of ρ_L with one relevant eigenperturbation. Comparison with Fig. 1 shows that Σ and ρ_L correspond to u and w, respectively.

two fixed-point potentials, $U^*_{\rm tr}(y)$ and $U^*_{SO}(y)$, which decay faster than y^{-2}: the trivial one, $U^*_{\rm tr}(y)\equiv0$, with one relevant eigenperturbation and a nontrivial one, $U^*_{SO}(y)$, at positive Σ, with no relevant eigenperturbation. They represent the critical delocalization transition and the delocalized phase, respectively.

Figure 2(a) also shows that the function $\rho_L(\Sigma)$ goes through the origin, $\Sigma=0$, because $U^*(y)\equiv0$ is always a solution of (5.2) and compatible with the boundary conditions. This must be valid for all τ. At the fixed point $U^*_{\rm tr}\equiv0$, the eigenvalue λ_1 satisfies $\lambda_1/\zeta=\tau-1=1$ for $\tau=2$. Since the linearized flow around $U^*_{\rm tr}\equiv0$ is equivalent to the linear RG, one recovers the value for λ/ζ as obtained within the linear scheme. The point $\Sigma=0$ is the only fixed point with a relevant eigenperturbation which decays faster than y^{-2}. Therefore the delocalization transition is governed by this fixed point for all potentials which decay faster than y^{-2}. In this case, the critical point is at potential strength, $G^*=0$ and the critical exponent $\nu_\parallel=2$.

The phase diagram which corresponds to Fig. 2(a) is plotted in Fig. 2(b). To demonstrate that it exhibits the

same topology as obtained from the exact transfer-matrix calculations, the amplitude Σ is plotted versus the part of the function ρ_L where λ_1/ζ is relevant.

Together with the preceding section one has for $\tau=2$ the following picture: all potentials which have only one minimum at $z=0$ and decay as $V(y)\sim y^{-r}$ with an exponent $r>1$ delocalize at $G^*=0$. If they decay faster than y^{-2}, the restriction to one minimum is not necessary. If they have more than one extremum and decay more slowly than y^{-2} and faster than y^{-1} one expects a delocalization transition with nontrivial phase boundaries as in the marginal case $r=2$, but in contrast to the marginal case, the phase transition is of first order for $W<0$. If the decay of the potential is slower than y^{-1} the long-range part of the potential governs the critical behavior and one has the phase boundary $W=0$.

If the potential has more than one extremum and decays with an exponent $r=2$ the marginal scaling regime predicts a nonuniversal behavior, with w dependent exponents $\nu_\parallel$; compare Sec. III. This should, in fact, apply to a whole range of τ until the function $\rho_L(\Sigma)$ develops more structure.

C. Bifurcation of fixed points

Now, let us increase the τ value, e.g., (i) to study solidlike membranes [30] with $\tau\simeq 4$ [15] or (ii) to investigate the delocalization of interfaces in $d=3$ which corresponds to $\tau=\infty$. For $\tau=\infty$, one has $\zeta=0$ and nontrivial fixed points are no longer compatible with the boundary conditions: for large $|y|$, the fixed-point potentials, $U^*(y)$, are governed by the linearized version of (5.2) but the linear solution exhibits oscillatory behavior in the limit of infinite τ and is therefore excluded [11].

The function $\rho_L(\Sigma)$ develops more structure as τ is increased. To see this, consider the eigenperturbations f_λ at the trivial fixed point $U^*_{\rm tr}$ which satisfy (5.3) with $[U^*(y)]''\equiv 0$. The two independent solutions of (5.3) can now be obtained explicitly for all y. One solution is given by

$$f^{(1)}_\lambda(y)=\exp(-y^2/4)\mathfrak{D}_{\tau-\lambda/\zeta-1}(y)/\mathfrak{D}_{\tau-\lambda/\zeta-1}(0)\ , \quad (5.7a)$$

where $\mathfrak{D}_\mu(y)$ denote parabolic cylinder functions and $\mathfrak{D}_\mu(0)=\Gamma(\frac{1}{2})2^{\mu/2}/\Gamma((1-\mu)/2)$, Ref. [32]. The second solution is given by

$$f^{(2)}_\lambda(y)=\frac{1}{N}f^{(1)}_\lambda(y)\int_{y_0}^{y}dx\, e^{-(1/2)x^2}/[f^{(1)}_\lambda(x)]^2\ , \quad (5.7b)$$

where N is a normalization factor. For large $|y|$, one has $f^{(1)}_\lambda(y)\sim|y|^{\tau-1-\lambda/\zeta}e^{-y^2/2}$; compare (5.5). It then follows from (5.7b) that $f^{(2)}_\lambda(y)\sim|y|^{-\tau+\lambda/\zeta}$ which is excluded by the condition $C_f=0$ in (5.6). In addition, all eigenperturbations have to be symmetric around the origin which implies $\partial f_\lambda(y)/\partial y|_0=0$. For the eigenfunction $f^{(1)}_\lambda(y)$ as given by (5.7a), one has

$$\left.\frac{\partial f^{(1)}_\lambda(y)}{\partial y}\right|_0=\mathfrak{D}'_{\tau-\lambda/\zeta-1}(0)/\mathfrak{D}_{\tau-\lambda/\zeta-1}(0)\ , \quad (5.8)$$

with $\mathfrak{D}'_\mu(0)=\Gamma(-1/2)2^{(\mu-1)/2}/\Gamma(-\mu/2)$, Ref. [32], and $\mu=\tau-\lambda/\zeta-1$. Thus $\partial f^{(1)}_\lambda(y)/\partial y|_0$ vanishes at the poles of $\Gamma(-\mu/2)$ which implies that

$$\tau-\lambda/\zeta-1=2k\ , \quad (5.9)$$

with k an arbitrary integer $k\geq 0$. In this case, the order of the parabolic cylinder function is even integer and the parabolic cylinder functions can be expressed in terms of the Hermite polynomials, $\mathfrak{H}_n(y)$:

$$f^{(1)}_\lambda(y)=\exp(-y^2/2)\mathfrak{H}_{2k}(y/\sqrt{2})/\mathfrak{H}_{2k}(0)\ . \quad (5.10)$$

The largest eigenvalue is $\lambda/\zeta=\tau-1$ corresponding to $k=0$ which was obtained both numerically, compare Fig. 2, and from the matching procedure in Sec. III. However, as τ is increased, additional relevant eigenperturbations appear at

$$\tau_{k+1}=2k+1 \quad \text{with } k=1,2,3,\ldots\ . \quad (5.11)$$

At $\tau=\tau_2=3$, one has $\lambda_1/\zeta=\tau_2-1=2$ and $\lambda_2/\zeta=\tau_2-1-2=0$; thus an additional marginal perturbation appears which becomes relevant for $\tau>\tau_2$. At $\tau=\tau_3=5$, another eigenvalue, λ_3/ζ, becomes relevant and one has $\lambda_1/\zeta=4$, $\lambda_2/\zeta=2$, and $\lambda_3/\zeta=0$. In general, one has k relevant eigenperturbations for $\tau>\tau_{k+1}$. This result is exact since all eigenperturbations are studied at the trivial fixed point $U^*_{\rm tr}$ and the RG is exact to first order in U.

This behavior of the eigenperturbations as a function of τ is related to the fixed-point function, $\rho_L(\Sigma)$. The linear RG has two fixed-point solutions one of which is

$$U^*_{\rm lin}(y)=\exp(-y^2/4)\mathfrak{D}_{\tau-1}(y)\ . \quad (5.12)$$

For $\tau-1=2k$, with integer $k\geq 0$, the function $U^*_{\rm lin}(y)$ as given by (5.12) is even and therefore compatible with the boundary conditions of delocalization. The solution (5.12) should be a good approximation in a small Σ interval around $\Sigma=0$. Since $U^*_{\rm lin}(y)$ has no long-ranged tail, the function $\rho_L(\Sigma)$ must be almost zero in an interval around $\Sigma=0$ which indicates that $\rho_L(\Sigma)$ has a turning point. Such turning points are present for $\tau=\tau_{k+1}$, with $k=1,2,3,\ldots$ (the case $\tau=1$ will be discussed in Sec. V D). Note that these are the same τ values as the above τ_k where the new λ_k/ζ becomes relevant. Out of every turning point bifurcates a new pair of extrema and a new pair of zeros of the function $\rho_L(\Sigma)$. This pair of extrema and zeros moves further apart as τ is increased. The zeros of $\rho_L(\Sigma)$ correspond to short-ranged fixed points. Between each such pair of short-ranged fixed points, the next turning point appears at the origin as τ reaches the next odd integer. This happens an infinite number of times as τ goes to infinity. The corresponding eigenvalues λ_k/ζ are positive and thus relevant in the interval between the two extrema and are irrelevant elsewhere.

Let us illustrate this for the case $\tau=4$, which corresponds to polymerized or solidlike membranes; see Fig. 3. One has one pair of extrema which has bifurcated from the turning point at $\tau=\tau_2=3$. There are two new short-ranged fixed points, $U^*_{S2}(y)$ and $U^*_{S\bar{2}}(y)$, with $\Sigma\simeq\Sigma_{S2}$ and

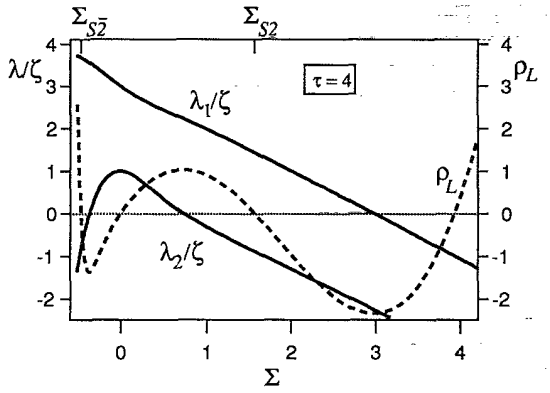

FIG. 3. The function $\rho_L=\rho_L(\Sigma)$ and the first two relevant eigenperturbations for $\tau=4$. Compared to $\tau=2$, the function ρ_L has a new pair of extrema which has bifurcated out of the trivial fixed point at $\tau=3$. Between these extrema, an additional eigenvalue is relevant. There are now two additional short-ranged fixed points, $U^*_{S\bar{2}}(y)$ and $U^*_{S2}(y)$, with amplitudes $\Sigma_{S\bar{2}}$ and Σ_{S2} which both have one relevant eigenvalue.

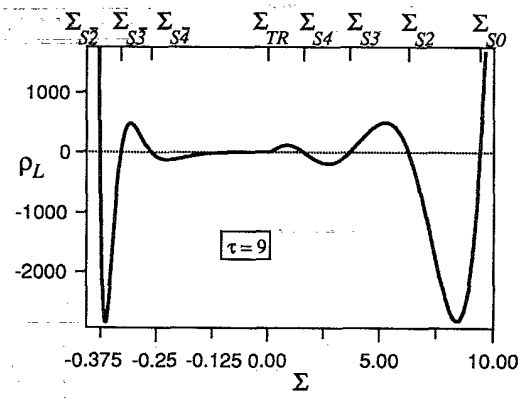

FIG. 5. The function $\rho_L=\rho_L(\Sigma)$ at $\tau=9$. Note that for $\Sigma<0$ the scale is magnified by a factor 10. Since τ is odd a turning point exists at the origin. One has three pairs of extrema which have bifurcated out of the origin at $\tau=3,5,7$, respectively. The trivial fixed point has now four relevant eigenperturbations.

$\Sigma=\Sigma_{S\bar{2}}$, respectively, as shown in Fig. 4. They both have only one relevant eigenperturbation. The trivial fixed point U^*_{tr} has two relevant eigenperturbations and corresponds to a multicritical point. It separates two different families of interaction potentials which have a critical behavior governed by $U^*_{S2}, U^*_{S\bar{2}}$.

The form of the potential $U^*_{S\bar{2}}$ and the value $\lambda_1/\zeta\simeq\tau$ at this fixed point implies that it corresponds to a first-order transition. The other fixed point represents a second-order transition and potentials governed by this fixed point have at least one minimum at $y>0$ and a maximum at $y=0$. Manifolds within such potentials will delocalize for a finite depth of the potential minima. This is a second class of delocalization transitions of solidlike membranes with $\tau=4$ in short-ranged potentials (compare the universality found for manifolds with $\tau=2$: all short-range potentials delocalize at potential strength, $G=0$).

The trivial fixed point U^*_{tr}, however, still governs the behavior of all potentials which have only one minimum at $y=0$ even though U^*_{tr} has the largest number of relevant eigenperturbations among all short-ranged fixed points, e.g., for $\tau=9$, U^*_{tr} has four relevant eigenperturbations and one has additional three pairs of short-ranged fixed points; see Fig. 5 for the whole function $\rho_L(\Sigma)$. Indeed, both U^*_{S2} and $U^*_{S\bar{2}}$ and all other new fixed points which occur at larger τ values have more than one extremum. However, in general one has classes of short-ranged potentials which delocalize via a first-order transition and another group of such potentials which exhibit multicritical delocalization behavior depending on the number of their extrema.

D. The limit of zero τ

It is interesting to investigate how a reduction of lateral dimension or a decrease of τ changes the critical

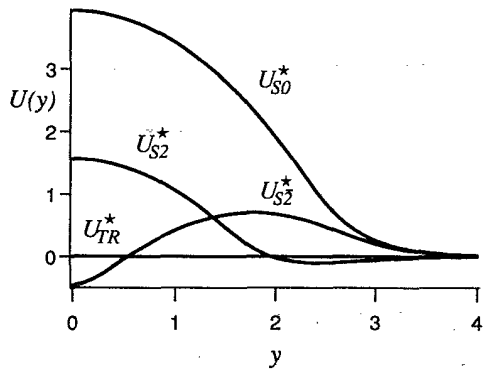

FIG. 4. The four short-ranged fixed-point potentials at $\tau=4$. Only the part for $y>0$ is shown. The fixed point U^*_{S0} is purely repulsive and has no relevant eigenperturbation. It represents the delocalized phase. The fixed points $U^*_{S\bar{2}}(y)$ and $U^*_{S2}(y)$ have one relevant eigenperturbation. The fixed point $U^*_{S\bar{2}}$ represents a first-order transition.

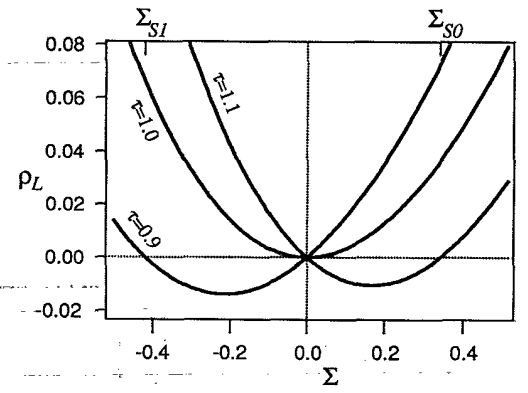

FIG. 6. The function $\rho_L=\rho_L(\Sigma)$ for $\tau\simeq1$. The minimum Σ_{min} of ρ_L is at $\Sigma>0$, $\Sigma=0$, and $\Sigma<0$ for $\tau>1$, $\tau=1$, and $\tau<1$, respectively. A relevant eigenvalue exists only for $\Sigma<\Sigma_{min}$. Thus one has a nontrivial short-ranged fixed point with one relevant eigenperturbation for $\tau<1$.

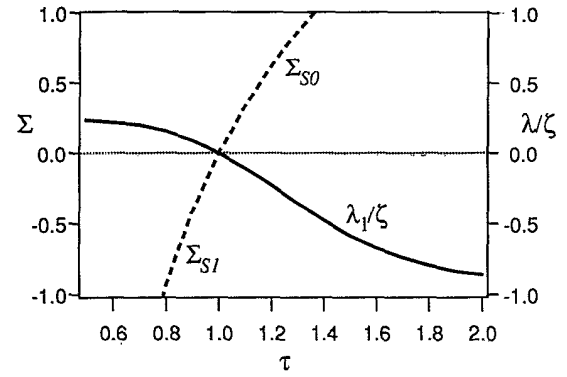

FIG. 7. Largest eigenvalue λ_1/ζ and amplitude Σ of the nontrivial fixed point with $\rho_L=0$ as a function of τ. This fixed point, denoted by $U_{S1}(y)$ and $U_{S0}(y)$, has the amplitudes Σ_{S1} and Σ_{S0} for $\tau<1$ and $\tau>1$; compare Fig. 6.

behavior at delocalization transitions. In general, one has an increase of thermal fluctuations which acts to overcome the attractive potential.

In Fig. 6, the function $\rho_L(\Sigma)$ is plotted for different values of decreasing τ. One can see a shift of the minimum and the zeros of $\rho_L(\Sigma)$ towards smaller Σ. At $\tau=\tau_1=1$, the minimum has reached the origin. For $\tau<1$, the minimum of $\rho_L(\Sigma)$ moves further to negative Σ values and one has again two short-ranged fixed points. The nontrivial fixed point is now characterized by $\Sigma=\Sigma_{S1}<0$ and will be denoted by $U^*_{S1}(y)$. The latter fixed point should represent a delocalization transition, while $U^*_{tr}(y)$ now describes the delocalized phase: thus the role of the two fixed points has been interchanged.

This can also be seen if one considers the eigenvalues at the short-ranged fixed points. For the nontrivial one, denoted by $U^*_{S1}(y)$ and $U^*_{S0}(y)$ for $\tau<1$ and $\tau>1$, respectively, the largest eigenvalue λ_1/ζ is plotted as a function of τ in Fig. 7. At $\tau=1$, the eigenvalue λ_1/ζ changes sign and becomes relevant for smaller τ. The τ dependence of λ_1/ζ for the trivial fixed point is simply given by $\lambda_1/\zeta=\tau-1$, as follows from (5.8) which also changes sign at $\tau=1$. Thus the boundary value $\tau_1=1$ can be simply deduced from the fact that the eigenperturbation f_{λ_1} of the trivial fixed point becomes marginal at $\tau=1$ and is irrelevant for $\tau<1$ [33,34]. So we conclude that systems with $\tau<1$ can undergo a delocalization transition at a finite potential strength.

VI. DELOCALIZATION OF INTERFACES IN THREE DIMENSIONS

Finally, consider the interfacial behavior in a three-dimensional system which contains a planar defect, such as a three-dimensional Ising model with a plane of weakened bonds. If the temperature T is above the roughening temperature, the delocalized interface is rough and the periodic lattice potential is again irrelevant [35]. In the latter situation, the effective potential for the interface arises only from the defect and will again be short ranged if the Ising model has only short-ranged coupling constants.

For $d<3$, the roughness exponent $\zeta=(3-d)/2$ and $\tau=(d-1)/\zeta=2(d-1)/(3-d)$. As $d=3$ is approached from below, the roughness exponent ζ goes to zero and τ goes to infinity. In this limit, infinitely many short-ranged fixed points appear around the critical fixed point as discussed in Sec. V C. But the most relevant perturbation with eigenvalue λ_1 has a finite limit in $d=3$, since $\lambda_1=\zeta(\tau-1)=2-3\epsilon/2$ for $d=3-\epsilon$. This implies the critical exponent

$$\nu_\parallel=1/\lambda_1=\tfrac{1}{2} \quad \text{for } d=3 \; . \tag{6.1}$$

This value is the mean-field value and should be valid for all dimensions $d>3$ since fluctuations are unimportant in the latter case. In $d=3$, on the other hand, one expects to find confluent logarithmic singularities. The nature of these logarithmic terms can be obtained by an appropriate modification of the matching procedure described in Sec. IV B. Instead of (4.7) and (4.6), one now has

$$\bar{a}_\perp^2 \sim \ln(b_0) \sim \ln\xi_\parallel \; . \tag{6.2}$$

The matching condition now reads

$$\frac{K}{\xi_\parallel^2(b_0)} \sim b_0^2 \bar{a}_\perp^{-2} \int_{-\infty}^{\infty} e^{-(1/2)u^2}(u^2-1)V(\bar{a}_\perp u) \; . \tag{6.3}$$

If one uses again the potential as defined in (4.13)

$$V(z)=G(z^2+z_0^2)^{-r/2} \tag{6.4}$$

the matching leads via (4.14) to

$$\frac{K}{\xi_\parallel^2(b_0)} \sim |G| b_0^2 [c_1 \ln(b_0)^{-3/2} + \ln(b_0)^{-[1+(r/2)]}] \tag{6.5}$$

where c_1 is independent of b_0. It follows that the divergence of $\xi_\parallel$ is given by

$$\xi_\parallel \sim \begin{cases} |G|^{-1/2}[\ln(1/|G|)]^{3/4} & \text{for } r>1 \\ |G|^{-1/2}[\ln(1/|G|)]^{(r/4)+(1/2)} & \text{for } r<1 \; . \end{cases} \tag{6.6}$$

VII. SUMMARY

The delocalization of low-dimensional manifolds in symmetric potentials was investigated. For interfaces in $d=1+1$, the critical behavior can be determined exactly by transfer-matrix methods. For potentials $V(z)$ with $V(z)\approx W/|z|^2$ for large $|z|$, one finds a complex phase diagram which has the same topology as for wetting transitions. Thus one again finds three subregimes, see Fig. 1: subregime (C) exhibits unusual first-order transitions, subregime (B) nonuniversal exponents depending on the amplitude of the power-law tail, see (3.17), and subregime (A) shows an essential singularity in the correlation length at the transition line.

For interfaces in $d=1+1$, the critical behavior can also be determined by an exact functional RG. This has been explicitly shown for wetting transitions [28,29]. In the present context, one again has a line of RG fixed points for subregime (B) and a separatrix of the RG flow for subregime (C).

In general, the critical behavior at delocalization tran-

sitions cannot be determined exactly unless the potential has a *single* minimum (at $z=0$). In the latter case, linear renormalization up to first order in $V(z)$ is reliable and the critical exponents can be determined by a matching procedure; see Sec. IV. As a result, we find that the critical behavior is universal, provided the potential $V(z)$ decays faster than $|z|^{-1}$. In the latter case, the critical exponent $\nu_\parallel$ is given by $\nu_\parallel=1/[\zeta(\tau-1)]$; see (4.12) and (4.15).

Potentials with *several* minima were treated within a nonlinear RG scheme which depends only on the parameter τ. One finds a similar sequence of bifurcations of the line of fixed points as in the case of unbinding or wetting transitions [6]. In contrast to unbinding, the trivial fixed point $U^*_{\rm tr}(0)\equiv 0$ is always an allowed solution compatible with the boundary conditions. Out of this trivial fixed point, new pairs of short-ranged fixed points bifurcate at odd values of τ. Simultaneously new relevant eigenperturbations occur; see Sec. V C. In the present paper, we have used only one particular RG transformation based on a hard-cutoff procedure. However, we have previously shown for unbinding transitions that a second functional RG based on a smooth-cutoff procedure led to the same sequence of bifurcations [6]. This should also apply to the delocalization phenomena studied here.

On the other hand, we do not have any evidence for the bifurcations apart from the procedure of functional renormalization. One case which could be studied explicitly by computer simulations is the delocalization of solidlike membranes in short-ranged potentials. As argued at the end of Sec. V C, one should find both first-order and second-order transitions depending on the shape of the external potential.

The critical exponent of the trivial fixed point in the limit of large τ is analytical: the critical exponent $\nu_\parallel$ reaches the mean-field value, $\nu_\parallel=\frac{1}{2}$, which was also found within a linear matching procedure in the marginal case with $\zeta=0$, or $\tau=\infty$. The critical behavior involves confluent logarithmic singularities; see (6.6). This critical behavior should be accessible to computer simulations for the three-dimensional Ising model.

Another interesting aspect of the present work is the critical delocalization behavior for $\tau<2$. In this case, short-ranged potentials $V(z)$ with $|V(z)|\ll 1/z^\tau$ for large z should exhibit *universal* critical exponents. For $\tau<1$, the critical behavior is governed by the trivial fixed point; for $\tau<1$, on the other hand, the interface or membrane delocalizes at a finite, negative potential strength. It would be interesting to find an exactly solvable model to confirm this result.

APPENDIX A: POTENTIALS WITH $1/z^2$ POWER-LAW TAILS IN $d=2$ DIMENSIONS

In this appendix, the phase boundary of the potential as given by

$$V(z)=\begin{cases} U & \text{for } -z_0<z<z_0 \\ W/z^2 & \text{for } |z|\geq z_0 \end{cases}, \tag{A1}$$

compare (3.10), will be determined.

To simplify the notation, it is convenient to introduce the rescaled variables: $y=z/z_0$, $u=2Kz_0^2U/T^2$, $w=2KW/T^2$ and rescaled energies: $e_n=2Kz_0^2E_N/T^2$. Then the Schrödinger-type equation becomes

$$-\phi_n''+v(y)\phi_n=e_n\phi_n \tag{A2}$$

with

$$v(y)=\begin{cases} u, & -1<|y|<1 \\ w/y^2, & |y|\geq 1 \end{cases}. \tag{A3}$$

The delocalization of interfaces corresponds in the transfer-matrix formalism to the transition from a bound state to a scattering state [22]. For the bound state one has $e_0=-|e_0|\equiv -q^2$. We are therefore interested in the limit of small q^2. In the interval $-1<y<1$, ϕ_0 has the form

$$\phi_0=\begin{cases} C\cosh[(u+q^2)^{1/2}y] & \text{for } u>0 \quad \text{(A4)} \\ C\cos[(|u|-q^2)^{1/2}y] & \text{for } u<0\,. \quad \text{(A5)} \end{cases}$$

If $|y|\geq 1$ one has

$$\phi_0=C'\sqrt{y}K_\mu(qy) \tag{A6}$$

with $\mu=(\frac{1}{4}+w)^{1/2}$ and $K_\mu(qy)$ is a modified Bessel function in standard notation. The matching condition of ϕ_0 and ϕ_0' at $|y|=1$ leads to

$$(u+q^2)^{1/2}\tanh[(u+q^2)^{1/2}]-\tfrac{1}{2}=L_\mu(q)\,, \tag{A7}$$

$$-(|u|-q^2)^{1/2}\tan[(|u|-q^2)^{1/2}]-\tfrac{1}{2}=L_\mu(q)\,, \tag{A8}$$

for $u>0$ and $u<0$, respectively, and

$$L_\mu(q)=q\frac{\partial K_\mu/\partial q}{K_\mu(q)}\,. \tag{A9}$$

These equations determine $q^2=-e_0$ as a function of the parameters u,w. The case $w\geq\frac{1}{4}$ (i.e., $\mu\geq 0$) will only be considered here. Using the asymptotic behavior of $K_\mu(x)$ for small q, we obtain the phase boundary of the delocalization transition from

$$\sqrt{u_c}\tanh\sqrt{u_c}-\tfrac{1}{2}=-(\tfrac{1}{4}+w_c)^{1/2}\,, \tag{A10}$$

$$-\sqrt{|u_c|}\tan\sqrt{|u_c|}-\tfrac{1}{2}=-(\tfrac{1}{4}+w_c)^{1/2}\,, \tag{A11}$$

again for $u>0$ and $u<0$, respectively. The solution of these equations is shown in Fig. 1. The case $w\leq\frac{1}{4}$ requires a rewriting of the matching condition (A7) and since this is described in Ref. [22] for wetting transitions it is not repeated here. As shown in Fig. 1 for $u\geq u_c=0.595$ the phase boundary is a line at $w=-\frac{1}{4}$.

APPENDIX B: MATCHING PROCEDURE FOR THE LONG-RANGED POTENTIAL WITH TWO PARAMETERS

The matching procedure of the potential as defined in (4.22) will here be presented in some detail to show how to get the critical exponents from this type of calculation.

We start with V as defined in (4.22):

$$V(z)=\begin{cases} U, & |z|<z_0 \\ W/z^r, & |z|>z_0 \end{cases} . \tag{B1}$$

In Sec. IV we derived the general expression

$$\frac{K}{\xi_{\parallel}^{2n}(b_0)}=b_0^{d-1}b_0^{2\zeta}\bar{a}_{\perp}^{-2}\int_{-\infty}^{\infty}dz'\frac{1}{\sqrt{2\pi}}\frac{1}{\bar{a}_{\perp}}e^{-(1/2)z'^2/\bar{a}_{\perp}^2} \times\left[\frac{z'^2}{\bar{a}_{\perp}^2}-1\right]V(z') . \tag{B2}$$

For the matching procedure, see (4.9). The change of the variables, $u\equiv z/\bar{a}_{\perp}$, leads to

$$\frac{K}{\xi_{\parallel}^{2n}(b_0)}\sim b_0^{d-1}\int_0^{\infty}du\, e^{-(1/2)u^2}(u^2-1)V(\bar{a}_{\perp}u) , \tag{B3}$$

where the relation $\bar{a}_{\perp}\sim b_0^{\zeta}$ as in (4.7) has been used.

The integral in (B3) is divided up into

$$I_S=2U\int_0^{z_0/\bar{a}_{\perp}}du\, e^{-(1/2)u^2}(u^2-1) \tag{B4}$$

and

$$I_L=2W\int_{z_0/\bar{a}_{\perp}}^{\infty}du\, e^{-(1/2)u^2}(u^2-1)u^{-r}\bar{a}_{\perp}^{-r} . \tag{B5}$$

For I_S, one obtains

$$I_S=U[-2z_0/\bar{a}_{\perp}+(z_0/\bar{a}_{\perp})^{-3}+O(\bar{a}_{\perp}^{-5})] . \tag{B6}$$

Further, I_L has a singularity from the lower bound where $\exp(-\frac{1}{2}u^2)(u^2-1)$ can be expanded around $u=0$. To extract the leading term we split I_L again into

$$I_{L1}=-2W\int_{z_0/\bar{a}_{\perp}}^{\infty}du\, u^{-r}\bar{a}_{\perp}^{-r} , \tag{B7}$$

and

$$I_{L2}=2W\int_{z_0/\bar{a}_{\perp}}^{\infty}du\,[e^{-(1/2)u^2}(u^2-1)+1]u^{-r}\bar{a}_{\perp}^{-r} . \tag{B8}$$

So we get for $r>1$

$$I_{L1}=-2W\frac{z_0^{1-r}\bar{a}_{\perp}^{-1}}{r-1} . \tag{B9}$$

To get the next-order term from I_{L2} one has to distinguish two cases: first it is assumed that I_{L2} has a singularity from the lower bound then the next-order term is

$$I_{L2}=3W\int_{z_0/\bar{a}_{\perp}}^{\infty}du\,[u^2+O(u^4)]u^{-r}\bar{a}_{\perp}^{-r} =3W\frac{z_0^{3-r}\bar{a}_{\perp}^{-3}}{r-3}+O(\bar{a}_{\perp}^{-r^<}) , \tag{B10}$$

with $r^<\equiv\min(r,5)$, valid for $r>3$.

In the other case, $1<r<3$, there is no singularity in I_{L2} and one gets

$$I_{L2}=3W\int_0^{\infty}du\,[e^{-(1/2)u^2}(u^2-1)+1]u^{-r}\bar{a}_{\perp}^{-r}+O(\bar{a}_{\perp}^{-3}) \equiv Wc_r^{<}\bar{a}_{\perp}^{-r}+O(\bar{a}_{\perp}^{-3}) . \tag{B11}$$

So the matching procedure gives finally

$$\frac{K}{\xi_{\parallel}^{2n}(b_0)}\sim b_0^{d-1}\left[\bar{a}_{\perp}^{-1}\left[-2z_0U-\frac{2Wz_0^{1-r}}{r-1}\right] +\bar{a}_{\perp}^{-3}\left[z_0^3U+\frac{3z_0^{3-r}W}{r-3}\right]+O(\bar{a}_{\perp}^{-r^<})\right] , \tag{B12a}$$

and

$$\frac{K}{\xi_{\parallel}^{2n}(b_0)}\sim b_0^{d-1}\left[\bar{a}_{\perp}^{-1}\left[-2z_0U-\frac{2Wz_0^{1-r}}{r-1}\right]+\bar{a}_{\perp}^{-r}Wc_r^{<} +O(\bar{a}_{\perp}^{-3})\right] , \tag{B12b}$$

for $r>3$ and $1<r<3$, respectively.

The leading term of (B12a) and (B12b) is

$$\frac{K}{\xi_{\parallel}^{2n}(b_0)}\sim b_0^{d-1}\bar{a}_{\perp}^{-1}\left[-U-\frac{Wz_0^{-r}}{r-1}\right] \sim b_0^{d-1-\zeta}\left[-U-\frac{Wz_0^{-r}}{r-1}\right] . \tag{B13}$$

The rescaling factor b_0 is chosen in such a way that $\xi_{\parallel}=b_0 a$, see (4.6), and from (B13) one has

$$\xi_{\parallel}\sim\left[\frac{-Wz_0^{-r}}{r-1}-U\right]^{-\nu_{\parallel}} , \tag{B14}$$

with

$$\nu_{\parallel}=\frac{1}{d-1-\zeta}=\frac{1}{\zeta(\tau-1)} . \tag{B15}$$

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