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Universal Aspects of Interacting Lines and Surfaces

Michael Lässig[(a)] and Reinhard Lipowsky[(a,b)]

[(a)]Institut für Festkörperforschung, Forschungszentrum, 52425 Jülich, Germany
[(b)]Max-Planck-Institut für Kolloid– und Grenzflächenforschung, Kantstr. 55, 14513 Teltow–Seehof, Germany

Contents

1 Interfaces, strings and membranes

The critical behavior of interfaces is related to their reduced dimensionality. [1] In some cases, the interface can simply be viewed as a planar 2–dimensional system. However, it can also 'escape' into the third dimension and then attain nonplanar morphologies. This *roughening* of the interface can be thermally excited or induced by frozen randomness. In addition, the interface has a certain depth profile and thus has itself a third dimension. This intrinsic thickness of the interface can become mesoscopic as in *wetting* phenomena [1, 2]: one then has a thin layer which is bounded by two interfaces. The thickening of this layer leads to the *unbinding* of these two interfaces, see fig. 1.

These critical effects are not restricted to three dimensions. Indeed, roughening, wetting and general unbinding phenomena also occur in 2–dimensional systems where they are governed by the behavior of 1–dimensional domain boundaries. [3] Since these domain boundaries are governed by a finite line tension, their statistical mechanics is intimately related to other 1–dimensional lines or strings such as (i) steps or ledges on crystal surfaces, (ii) stretched (or directed) polymers, and (iii) vortex lines in superconductors.

It turns out that these 1–dimensional strings have scaling properties which are very similar to those of 2–dimensional membranes, i.e., thin sheets of molecules. [4] The most prominent examples of such membranes are bilayers of amphiphilic molecules which represent model systems for the rather complex membranes of biological systems. [4, 5] These membranes are also roughened by thermally–excited shape fluctuations. In addition, the adhesion and unbinding of membranes can be understood in close analogy to interfacial wetting, see fig. 1. Adsorption–desorption transitions of polymers [6] are a related unbinding phenomenon.

The unbinding of strings and surfaces is often driven by their shape fluctuations which *renormalize* their direct interaction arising from intermolecular forces. For thermally–excited fluctuations, this renormalization acts to increase the repulsive part of the interaction. At low temperatures, these fluctuations are weak and the renormalized interaction closely resembles the direct interaction. However, as the temperature T is increased, the renormalization becomes more and more effective up to a characteristic unbinding temperature, $T = T_*$, at which the manifolds undergo an transition from a bound to an unbound state.

The critical behavior at these unbinding transitions involves several diverging length scales such as the *mean separation* or the *roughness* of these manifolds . In addition, other quantities such as the *probability for local contacts* are also singular at these transitions. This quantity represents a convenient starting point for a systematic field-theoretic treatment of these transitions [7].

This article reviews recent theoretical work on these critical phenomena

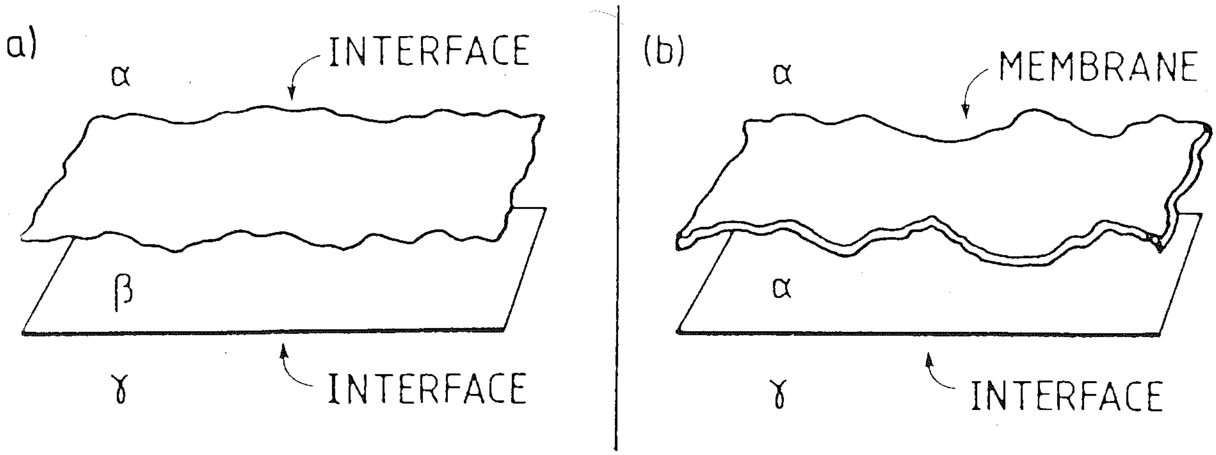

Fig. 1. (a) Wetting layer of phase β between two bulk phases α and γ. The thickening of this layer corresponds to the unbinding of the two interfaces bounding the layer; and (b) Adhesion of a flexible membrane consisting of a thin layer of molecules towards another interface. The shape fluctuations of the membrane act to unbind the two surfaces.

[7-14]. It is organized as follows. The scaling behavior of interacting manifolds and theoretical models for these systems are briefly reviewed in Sect. 2 and Sect. 3, respectively. Then, in Sect. 4, a new scaling picture is described [12] in which the probability of *local contacts* between the interacting manifolds plays a prominent role. This scaling picture is developed in some detail for the case of strings interacting via short–ranged or long–ranged potentials. In this context, we discuss in Sect. 4.7 the unbinding transition of bundles of nonintersecting strings in the transfer matrix approach [8, 9, 10, 13]. Sect. 4.8 contains extensions of the scaling picture to other systems.

The scaling picture can be justified in a systematic way by applying continuum field theory to these systems [7]. The central idea is to treat all interactions as *local operators* which form an operator algebra characterizing their universal short-distance properties. In this article, we do not assume any knowledge of field-theoretic renormalization; Sect. 5 can also be read as a self-contained introduction to some ideas in this subject. In Sect. 6 we apply these methods to a number of more difficult problems in the context of interfaces [7, 11, 14]: strings with long-ranged interactions, systems of many strings (which may be "bosonic" or "fermionic"), and interfaces of general dimensionality. In all cases, we find a number of nontrivial universality classes. We conclude this section with a brief outlook.

2 Scaling behavior of interacting manifolds

In this section, we will introduce the various quantities which are singular at roughening, wetting and unbinding transitions and define the corresponding critical exponents.

2.1 Roughness exponent ζ

As mentioned, low–dimensional manifolds are often rough, i.e., they make large transverse excursions from their mean or average position. More precisely, such a manifold is rough if the typical size, $\xi_\perp$, of its transverse excursions grows with its lateral size, $\xi_\parallel$. This behavior can usually be described by the scaling law

$$\xi_\perp \sim \xi_\parallel{}^\zeta \tag{2.1}$$

which defines the *roughness* exponent ζ. For interfaces and domain boundaries, the universality classes for this exponent are primarily determined by the *symmetry* of the two bulk phases adjacent to the interface.

2.2 Roughening exponent $\nu_\perp$

The roughness of the manifold can change as a function of temperature or some other control parameter. For example, the 2–dimensional interface between a periodic crystal and its vapor is smooth at low temperatures T with $\xi_\perp$ confined by the lattice potential. As the temperature is increased, this confining potential becomes less and less effective and the interface becomes delocalized up to a critical temperature T_* at which it undergoes a roughening or delocalization transition. For $T > T_*$, the interface is rough at all scales. Likewise, a domain boundary may be localized at low temperature by a defect line (in $D = 2$) or by a defect plane (in $D = 3$) but may become delocalized at sufficiently high temperatures.

As the critical temperature, T_*, is approached from below, the roughness $\xi_\perp$ typically grows as

$$\xi_\perp \sim 1/(T_* - T)^{\nu_\perp} \tag{2.2}$$

which defines the *roughening exponent* $\nu_\perp$. In general, this exponent depends on the nature of the effective potential confining the interface and one must distinguish several universality classes or scaling regimes of these potentials.

2.3 Unbinding exponent ψ

Now consider two interacting manifolds with local separation l. If these manifolds undergo an unbinding transition, their mean separation $\langle l \rangle$ diverges and grows as

$$\ell \equiv \langle l \rangle \sim 1/(T_* - T)^{\psi} \tag{2.3}$$

which defines the unbinding exponent ψ. This exponent in general also depends on the nature of the interaction potential experienced by the two manifolds.

In most cases of interest, the manifolds become rough or delocalized as they unbind. In fact, we will be primarily interested in situations in which the unbinding is driven by the roughening of the manifolds. One may then consider the roughness of the local separation field l which is defined by $\xi_\perp \equiv \langle [l - \langle l \rangle]^2 \rangle^{1/2}$. If the unbinding is driven by the shape fluctuations, one has $\xi_\perp \sim \ell$, and the unbinding exponent ψ is equal to the roughening exponent $\nu_\perp$ (there is one exceptional case where short–ranged attractive potentials compete with repulsive potentials of 'intermediate range', see Sect. 4.6).

2.4 Contact exponent ζ_0

Another quantity which exhibits singular behavior at roughening and unbinding transitions is the probability $\mathcal{P}_b$ of locally bound segments, i.e., of local contacts between the interacting manifolds. As discussed in some detail in the following sections, this quantity is quite generally given in terms of the one-point function of a local operator Φ and vanishes as

$$\mathcal{P}_b \sim \langle \Phi \rangle \sim \xi_{\parallel}{}^{-\zeta_0} \sim \xi_\perp{}^{-\zeta_0/\zeta} \quad . \tag{2.4}$$

close to criticality [7] * This defines the contact exponent ζ_0.

The simplest situation is exemplified by an interface characterized by a Gaussian probability distribution $\exp(-l^2/2\xi_\perp^2)/\xi_\perp$ for the fluctuating field l. In this case, we have $\mathcal{P}_b \sim 1/\xi_\perp$, and hence the contact exponent ζ_0 equals ζ.

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*A precise definition of "locally bound segments" must distinguish between "bosonic" and "fermionic" systems, see eqns. (5.4) and (6.26) below. The exponent ζ_0 will be called x in Sect. 5.

In general, two cases must be distinguished. If the attractive part of the potential is sufficiently short–ranged, the exponent ζ_0 is related to the roughening exponent $\nu_\perp$ via a scaling relation, see the relations (4.19) and (4.27) below. On the other hand, if the bound state of the manifold is controlled by an attractive potential which is sufficiently long-ranged, the exponent $\nu_\perp$ is determined by this long–ranged potential. The the exponent ζ_0 may still governed by a repulsive short–ranged potential, in which case the two exponents are independent.

3 Effective models for interacting manifolds

In the continuum limit, the position of each fluctuating manifold can be described by a displacement field $l = l(s)$ where s is a $d_\|$–dimensional coordinate parallel to a reference plane. For roughening or delocalization phenomena, the field l gives the distance of the manifold from this reference plane; for the unbinding of two interacting manifolds, this field measures the separation of these two manifolds.

The *effective Hamiltonian* for the displacement field l has the generic form [1]

$$\mathcal{H}\{l\} = \mathcal{H}_0\{l\} + \int \mathcal{V}[l(s)]d^{d_\|}s \tag{3.1}$$

where $\mathcal{H}_0\{l\}$ represents the elastic energy of the shape fluctuations in the completely unbound state and $\mathcal{V}(l)$ is an effective potential which acts to localize these shape fluctuations. $\mathcal{H}$ will also be called the *action* for the field l in order to avoid confusion with the corresponding Hamilton operator $\hat{H}$, the infinitesimal generator of the transfer matrix.

For roughening and delocalization phenomena, the potential $\mathcal{V}(l)$ can describe the effect of an underlying lattice which may be periodic [15] or quasi–periodic [16] Below, we will discuss the influence of a defect line (or defect plane) which acts to localize the interface. In this latter case, the potential $\mathcal{V}(l)$ is taken to be symmetric and to have local minima at the position of the defect and for $l = \pm\infty$. [7]

For wetting and adhesion phenomena, the potential $\mathcal{V}(l)$ describes the interaction energy of the two manifolds at separation l. Usually, the two manifolds cannot intersect one another, and this interaction potential contains a hard wall at $l = 0$ which ensures that the displacement field satisfies

$l \geq 0$.

In Sections 4.7 and 6.2, we will also study the case of many interacting lines and thus of a large number of displacement fields.

4 A refined scaling picture for unbinding phenomena

A localized manifold can be regarded as an ensemble of essentially uncorrelated humps, see fig. 2 below. This view leads to the concept of a fluctuation–induced interaction $\mathcal{V}_{fl}$ between the manifolds. In the case of thermally–excited fluctuations, $\mathcal{V}_{fl}$ represents the loss of entropy arising from the confinement. This fluctuation–induced interaction can be used in a heuristic way in order to understand the critical behavior at unbinding transitions.

It has been previously emphasized that a simple superposition of $\mathcal{V}_{fl}$ and the direct interaction $\mathcal{V}(l)$ does not predict the correct critical behavior unless the interaction $\mathcal{V}(l)$ is sufficiently long–ranged. [17] Here, a refined scaling picture [12] based on a *two–state model* for the interacting segments of the manifolds is described which is appropriate for any type of interaction potential. This scaling picture can be justified in a systematic way in the field-theoretic framework [7] described in Sections 5 and 6 below. A crucial role is played by the probability that two segments of the interacting manifolds form locally bound pairs.

In the following section, the refined scaling picture will be first described for 1–dimensional strings governed by a finite line tension. The case of *wetting in two dimensions*, i.e., of two strings in $D = 1 + 1$ dimensions interacting via general pair potentials is discussed in some detail. For the special case of attractive square–well potentials, similar scaling ideas have been previously formulated for the so–called reflection model in Ref. [18]. The extension of the refined scaling picture to other types of manifolds is briefly described in Sect.4.7 and 4.8. The same picture can be formulated for bundles and bunches of N manifolds where it leads to a N–state model. [12]

Now consider two interacting strings in $1+1$ dimensions with line tensions σ_1 and σ_2, respectively. The action (or effective Hamiltonian) for their local separation (or relative displacement field) l is given by

$$\mathcal{H}\{l\} = \int \{ \frac{1}{2}\sigma(dl/ds)^2 + \mathcal{V}[l(x)] \} ds \tag{4.1}$$

with the reduced line tension $\sigma = \sigma_1\sigma_2/(\sigma_1+\sigma_2)$. If one string has an infinite stiffness, say $\sigma_2 = \infty$, corresponding to a straight rigid boundary, one has $\sigma = \sigma_1/2$.

This model can be analysed in much detail by transfer matrix methods, and one can obtain the exact critical behavior for many potentials $\mathcal{V}(l)$. From these latter results, one knows that there are several universality classes for the unbinding transition which depend on the long–ranged part of the interaction potentials. As shown below, the refined scaling picture is valid for all of these universality classes.

4.1 Two–state model for interacting strings

In general, the direct interaction potential $\mathcal{V}(l)$ will contain a short–ranged part and a longer–ranged part, which will be denoted by $\mathcal{V}_b(l)$ and $\mathcal{V}_{ub}(l)$, respectively (the indices b und ub will become clear in a moment). It will be convenient to introduce a microscopic length scale l_b and to define these two parts of the potential via

$$\mathcal{V}_b(l) \equiv \mathcal{V}(l) \qquad \text{for} \quad l < l_b \tag{4.2}$$

and

$$\mathcal{V}_{ub}(l) \equiv \mathcal{V}(l) \qquad \text{for} \quad l > l_b \quad , \tag{4.3}$$

respectively.

If the interaction potential contains an *attractive* short–ranged part, the scale l_b is given by the potential range of this attractive part. If the short–ranged potential is purely repulsive, the choice of l_b is somewhat arbitrary but it should be small compared to the length scales which enter the long–ranged part. In any case, the short–ranged part will contain the hard wall interaction which ensures that the two manifolds cannot intersect.

Two string segments which interact via such a potential can attain two different local states, see fig. 2: (i) They are locally unbound if their separation exceeds the length scale l_b; and (ii) They form a locally bound pair if

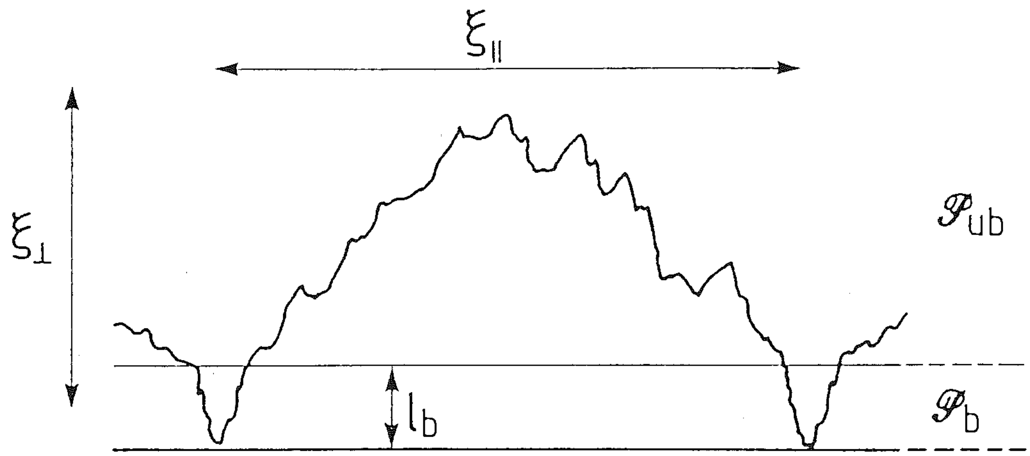

Fig. 2. The bound state of two strings consists of humps which have lateral and transverse extension $\xi_{\parallel}$ and $\xi_{\perp}$, respectively. The microscopic scale l_b represents the range of the short–ranged part of the interaction potential. Two adjacent segments of the two strings are locally bound and unbound with probability $\mathcal{P}_b$ and $\mathcal{P}_{ub}$, respectively.

their separation is smaller than l_b. The probabilities for these two different local configurations will be denoted by $\mathcal{P}_{ub}$ and $\mathcal{P}_b$, respectively.

Note that even if the string segments are locally unbound, they still have a finite separation. Therefore, compared to the situation in which the strings are completely separated, locally unbound segments have an excess free energy ΔF_{ub} per unit length. Likewise, the excess free energy per unit area of a locally bound pair will be denoted by ΔF_b. Thus, the excess free energy per unit length of the two strings can be estimated as

$$\Delta F = \Delta F_{ub}\,\mathcal{P}_{ub} + \Delta F_b\,\mathcal{P}_b \quad . \tag{4.4}$$

If the unbinding transition is continuous, both the excess free energy ΔF_{ub} of the locally unbound segments and the probability $\mathcal{P}_b$ for locally bound pairs must vanish in a continuous way whereas $\mathcal{P}_{ub} = 1 - \mathcal{P}_b \approx 1$ as the transition is approached. In addition, all critical quantities should scale with a single length scale which is here taken to be the roughness $\xi_{\perp}$ of the string separation. Since $\xi_{\perp}$ diverges at the transition, one anticipates that both ΔF_{ub} and $\mathcal{P}_b$ scale as inverse powers of $\xi_{\perp}$.

The excess free energy ΔF_b for bound segments, on the other hand, arises from configurations which have a separation of the order of the microscopic length scale l_b and, thus, will not depend on the diverging scale $\xi_{\perp}$.

4.2 Excess free energy of locally unbound segments

As indicated in fig. 2, the fluctuations in the string separation l can be regarded as an ensemble of humps which have the typical height $\xi_\perp$ and the longitudinal extension $\xi_\parallel$. Since the string separation is governed by a line tension, it diffuses like a (directed) random walk. Therefore, the two length scales $\xi_\perp$ and $\xi_\parallel$ satisfy the scaling relation

$$\xi_\perp \sim (T/\sigma)^{1/2}\, {\xi_\parallel}^{1/2} , \tag{4.5}$$

i.e., the roughness exponent has the value $\zeta = 1/2$.

Each hump of longitudinal and perpendicular extension $\xi_\parallel$ and $\xi_\perp$ has the volume $\mathcal{V} \simeq \xi_\parallel \xi_\perp$. Assuming that these humps are essentially uncorrelated and using the ideal gas law $P\mathcal{V} = T$ for a single degree of freedom together with the relation (4.5), one arrives at the pressure $P \sim T^2/\sigma{\xi_\perp}^3$. Alternatively, one may allude to the equipartition theorem and postulate that each such hump has a free energy $\sim T$. This implies that the hump free energy per unit (projected) area behaves as

$$\mathcal{V}_{fl}(\xi_\perp) \sim T/\xi_\parallel \sim T^2/\sigma\xi_\perp^2 \qquad \text{for large} \quad \xi_\perp \quad . \tag{4.6}$$

The disjoining pressure is now obtained from $P = \partial\mathcal{V}_{fl}/\partial\xi_\perp$. This estimate of the excess free energy of interacting strings is implicit in the work of Gruber and Mullins on steps or ledges on crystal surfaces [19] and has been explicitly derived by Prokovsky and Talapov for commensurate–incommensurate transitions in two dimensions [20].

Thus, the locally unbound segments suffer a loss of entropy per unit length which is given by $\mathcal{V}_{fl}(\xi_\perp)$ and which represents one contribution to the excess free energy ΔF_{ub}. In addition, these segments also have an interaction energy $\mathcal{V}_{ub}(l)$. It is plausible to assume and it can be checked a posteriori that the mean separation $\ell \equiv \langle l \rangle$ is proportional to $\xi_\perp$ close to the transition. In such a situation, the excess free energy of the locally unbound segments can be estimated as

$$\Delta F_{ub} = c_1 T^2/\sigma\xi_\perp^2 + \mathcal{V}_{ub}(c_2\xi_\perp) \quad . \tag{4.7}$$

Thus, for all long–ranged interactions which decay faster than $\sim 1/l^2$ for large l, one has $\Delta F_{ub} \sim 1/\xi_\perp^2$.

4.3 Scaling form for the probability distribution

Next, let us address the dependence of the probability $\mathcal{P}_b$ for locally bound segments on the roughness scale $\xi_\perp$. In general, the probability distribution $\mathcal{P}(l)$ for the string separation l should have the scaling form

$$\mathcal{P}(l) \approx \Omega(l/\xi_\perp)/\xi_\perp \qquad \text{for} \quad l \gg l_b \tag{4.8}$$

where the explicit factor $1/\xi_\perp$ arises from normalization. The probability $\mathcal{P}_b$ can then be estimated as

$$\mathcal{P}_b \simeq l_b \mathcal{P}(l_b) = (l_b/\xi_\perp)\Omega(l_b/\xi_\perp) \quad . \tag{4.9}$$

Therefore, the probability $\mathcal{P}_b$ is determined by the behavior of $\Omega(s)$ for small s.

For strings in two dimensions, the probability distribution $\mathcal{P}(l)$ is given in terms of the ground state wavefunction of the transfer matrix operator which can be explicitly calculated for many potentials. The results of these calculations will be described in the following subsections 4.5 and 4.6. In all cases, one finds the scaling behavior

$$\Omega(s) \sim s^{2\zeta_0 - 1} \qquad \text{for small } s \qquad \text{with } \zeta_0 > 0 \tag{4.10}$$

provided (i) the transition is continuous and (ii) the mean separation $\ell \sim \xi_\perp$ as assumed here (exceptions occur for interaction potentials with a long–ranged *repulsive* part which decays to zero not faster than $\sim 1/\xi_\perp^2$). The relation (4.10) implies that the probability for locally bound segments behaves as

$$\mathcal{P}_b \sim 1/{\xi_\perp}^{2\zeta_0} \sim 1/{\xi_\parallel}^{\zeta_0} \quad . \tag{4.11}$$

4.4 Competition between locally bound and unbound segments

Now, we can insert the two relations (4.11) and (4.7) into the expression (4.4) for the excess free energy in order to arrive at the estimate

$$\Delta F \approx c_1 T^2/\sigma\xi_\perp^2 + \mathcal{V}_{ub}(c_2\xi_\perp) + c_3 \Delta F_b/{\xi_\perp}^{2\zeta_0} \tag{4.12}$$

in the limit of large $\xi_\perp$ where c_1, c_2 and c_3 are dimensionless and positive coefficients.

The continuous unbinding of the two strings now correponds to a minimum of ΔF with respect to $\xi_\perp$ which goes continuously to infinity as the temperature or some interaction parameters are varied and the unbinding transition is approached.

Since the first term in (4.12) is positive, either the second or the third term must be negative in order to have a minimum of ΔF at a finite value of $\xi_\perp$, i.e., in order to have a bound state of the two strings. The third term, $\Delta F_b\ \mathcal{P}_b$, for locally bound segments, involves the exponent ζ_0 which has not been specified so far. A balance of this term with the two other terms in (4.12) shows that two cases must be distinguished: (i) For $0 < \zeta_0 < 1$, the third term dominates provided the long–ranged part $\mathcal{V}_{ub}(l)$ does not decay more slowly than $\sim 1/l^2$. The unbinding transition then occurs as ΔF_b goes to zero from below, and the critical behavior of $\xi_\perp$ is directly related to the probability $\mathcal{P}_b$; and (ii) For $1 < \zeta_0 < \infty$, a bound state is only possible for an attractive interaction $\mathcal{V}_{ub}(l) < 0$ which grows for large l or decays more slowly than $\sim 1/l^2$. In this case, unbinding occurs as this long–ranged attractive part goes to zero, and the corresponding critical behavior of $\xi_\perp$ is not affected by the probability $\mathcal{P}_b$.

4.5 Strong–fluctuation regime

The so–called strong–fluctuation regime consists of all interaction potentials which decay faster to zero than $\sim 1/l^2$ for large l. First, consider the case of an *attractive* square–well potential of depth $|U|$ and range l_b. In this case, the longer–ranged part $\mathcal{V}_{ub}$ is identically zero. The excess free energy for bound pairs can be estimated as $\Delta F_b \simeq -|U| + cT^2/\sigma {l_b}^2$ where the first and the second term represent the interaction energy and the entropy loss within the square well, respectively. In addition, transfer matrix calculations show that the probability distribution $\mathcal{P}(l)$ exhibits the scaling form (4.8) with $\Omega(s) \sim e^{-s}$. Since $\Omega(s) \sim$ const for small s, one has

$$\zeta_0 = 1/2 \qquad \text{and} \quad \mathcal{P}_b \sim 1/\xi_\perp \quad . \tag{4.13}$$

If these expressions are inserted into the excess free energy ΔF, minimization with respect to $\xi_\perp$ leads to

$$\xi_\perp \sim 1/|\Delta F_b|^{\nu_\perp} \qquad \text{with} \quad \nu_\perp = 1 \tag{4.14}$$

which is indeed the correct critical behavior for the unbinding transition within a square–well potential. [21, 22]

The same critical behavior applies to all potentials within the strong–fluctuation regime, i.e., to all potentials with a tail $\mathcal{V}_{ub}(l)$ which decays faster than $\sim 1/l^2$. [23] This can be understood by inspection of the expression (4.12) for the excess free energy: the tail $\mathcal{V}_{ub}(c\xi_\perp)$ is irrelevant compared to the short–ranged part $\Delta F_b \, \mathcal{P}_b \sim \Delta F_b/{\xi_\perp}^{2\zeta_0}$ provided the exponent ζ_0 still has the value $\zeta_0 = 1/2$. Thus, these interaction potentials are short–ranged even though there tails decay as inverse powers.

On the other hand, if the short–ranged potential is *repulsive*, field–theoretic renormalization group calculations [7, 11], described in Sect.5 below, and exact transfer matrix calculations [13] yield

$$\zeta_0 = 3/2 \qquad \text{and} \quad \mathcal{P}_b \sim 1/{\xi_\perp}^3 \quad . \tag{4.15}$$

Since this probability decays faster than the entropy loss $\sim 1/\xi_\perp^2$ and since ΔF_b is always positive, the term $\Delta F_b \, \mathcal{P}_b$ does not affect the minimum of ΔF and thus does not affect the critical behavior of $\xi_\perp$.

For example, one may confine the strings by a potential $\mathcal{V}_{ub}(l) \sim l$ or $\sim l^2$ and consider the limit in which the amplitude of such a potential goes to zero. In fact, the same value $\zeta_0 = 3/2$ applies to all attractive long–ranged potentials which decay more slowly than $\sim 1/l^2$. The marginal case $\mathcal{V}_{ub} \sim 1/l^2$, on the other hand, is more complex since it leads to a nonuniversal value for ζ_0 as discussed in the next subsection.

4.6 Intermediate fluctuation regime

Now, consider interaction potentials which behave as

$$\mathcal{V}(l) \approx W/l^2 \text{ for large } l \tag{4.16}$$

which defines the so–called intermediate fluctuation regime. [24] In this case, the behavior of the probability distribution $\mathcal{P}(l) = \Omega(l/\xi_\perp)/\xi_\perp$ for small l depends explicitly on the dimensionless parameter

$$w = 2\sigma W/T^2 \tag{4.17}$$

where σ and T are the string tension and the temperature, as before.

In fact, when considered as a function of w, the exponent ζ_0 has two branches depending on the sign of the short–ranged part of the interaction potential. For *attractive* short–ranged potentials, the probability distribution can be calculated using the results of Ref. [24]. As a result, one finds the singular behavior

$$\Omega(s) \sim s^{2\zeta_0 - 1} \qquad \text{with} \quad \zeta_0 = 1 - \sqrt{w + 1/4} \tag{4.18}$$

for small s. If these values for ζ_0 are used in the expression for ΔF, minimization of this excess free energy leads to

$$\xi_\perp \sim 1/|\Delta F_b|^{\nu_\perp} \qquad \text{with} \quad \nu_\perp = 1/(2 - 2\zeta_0) \tag{4.19}$$

which is again the correct critical behavior at the unbinding transitions. The case of a short–ranged potential is recovered for $w = 0$ with $\zeta_0 = 1/2$ and $\nu_\perp = 1$. Thus, the roughening exponent $\nu_\perp$ and the contact exponent ζ_0 are not independent but satisfy a scaling relation in this case.

The critical behavior as given by (4.19) applies to $-1/4 < w < 3/4$. For $w < -1/4$, the attractive potential $\mathcal{V}_{ub}(l) \sim 1/l^2$ is so strong, that the two strings cannot unbind. For $w > 3/4$, on the other hand, one has a relatively large potential barrier, and one then enters the so–called subregime (C) in which the probability distribution $\mathcal{P}(l)$ has the scaling from $\mathcal{P}(l) = (l_b/\xi_\perp)^{2\mu-1}\,\Omega(l/\xi_\perp)$ with $\mu \equiv \sqrt{w + 1/4}$ and $\Omega(s) \sim s^{1-2\mu}$ for small s. [25] This implies that the probability distribution $\mathcal{P}(l)$ attains the limiting form $\mathcal{P}(l) \sim l^{1-2\mu}$ and $\mathcal{P}_b = \mathcal{P}(l_b) \approx$ const at the unbinding transition.

The interactions within the intermediate fluctuation regime have also been studied by functional renormalization. [26, 27, 28] As a result, one finds a parabolic renormalization group flow with a whole line of fixed points. The fixed point line has two branches corresponding (i) to critical wetting transitions in the presence of attractive short–ranged potentials, and (ii) to completely wet states for *repulsive* short–ranged potentials. The latter branch is governed by completely repulsive fixed points at which the short–ranged potentials represent irrelevant perturbations. The corresponding scaling index depends on w. In two dimensions, one may set up an exact functional renormalization group (RG) in which the transfer matrix is diagonalized in an iterative manner. The scaling indices obtained from numerical iterations of this RG transformation [27] imply the contact exponent

$$\zeta_0 = 1 + \sqrt{w + 1/4} \quad . \tag{4.20}$$

The case of short–ranged potentials is again recovered for $w = 0$ with $\zeta_0 = 3/2$. Thus, if the strings experience an effectively repulsive short–ranged potential, the probability for local contacts is more and more suppressed with increasing w or W.

It is interesting to note that the intermediate fluctuation regime characterized by the contact exponents as given in (4.19) and (4.20) also applies to several other string systems if one makes an appropriate identification of the parameter w. First of all, the same critical behavior is found if the interaction potential $\mathcal{V}(l)$ is symmetric and does not contain a hard wall. [29] Such a potential would arise, e.g., for a domain boundary with stiffness σ_1 which interact with a plane of defects. In this case, w is still given by (4.17) with $\sigma = \sigma_1/2$.

Secondly, two strings in $D = 1 + d_\perp$ dimensions interacting with *short–ranged* interaction potentials belong to this intermediate regime. In this case, one has $w = (d_\perp - 3)(d_\perp - 1)/4$ [25] and thus

$$\zeta_0(d_\perp) = 1 \pm |d_\perp - 2|/2 \quad . \tag{4.21}$$

Thus, there are two branches for the contact exponent with $\zeta_0 = d_\perp/2$ and $\zeta_0 = 2 - d_\perp/2$, respectively, which cross at $d_\perp = 2$. The branch with $\zeta_0 = d_\perp/2$ represents effectively Gaussian fluctuations with $\mathcal{P}(l) \sim \exp[-l^2/2{\xi_\perp}^2]/{\xi_\perp}^{d_\perp}$.

For $1 \leq d_\perp \leq 2$, the critical unbinding transition and the completely unbound state are characterized by $\zeta_0 = d_\perp/2$ and by $\zeta_0 = 2 - d_\perp/2$, respectively. For $d_\perp \geq 2$, the two branches have exchanged and the critical unbinding transition now corresponds to $\zeta_0 = 2 - d_\perp/2$. The latter value is valid up to $d_\perp = 4$; for $d_\perp > 4$, one has $w > 3/4$ and thus enters the so–called subregime (C) as explained above.

Thirdly, the necklace models that we discuss in Sect. 4.7 can also be mapped onto the intermediate fluctuation regime.

4.7 Necklace models and nonintersecting strings

Consider the necklace model for three strings with line tensions σ_1, σ_2 and σ_3 [30, 8]: the strings interact via a hard-wall pair potential, which ensures that they cannot intersect, and via a short-ranged attractive 3–body force. Thus, the two outer strings experience the 3–body force and an effective repulsion arising from the confinement of the interior string. The entropy loss of the interior string behaves as $\sim 1/l^2$ which implies that the effective repulsion between the two outer strings scales in the same way.

This necklace model is characterized by the parameter $w = (\pi/\theta)^2 - 1/4$ with $\tan(\theta) = \sqrt{(\sigma_2/\sigma_1) + (\sigma_2/\sigma_3) + (\sigma_2{}^2/\sigma_1\sigma_3)}$ and $0 \leq \theta \leq \pi/2$. [8] It then follows from (4.18) and (4.19) that $\zeta_0(3) = 1 \pm \pi/\theta$. The minus sign corresponds to unbinding transitions in the presence of an effectively attractive 3–body force. However, since the minus sign leads to $\zeta_0 < 0$ corresponding to $w > 3/4$, these transitions belong to subregime (C) for which $\zeta_0 = 0$. On the other hand, if the short–ranged 3–body force between the three strings is effectively repulsive, this system is characterized by the contact exponent

$$\zeta_0(3) = 1 + \pi/\theta \quad . \tag{4.22}$$

Thus, if one keeps the three strings together by an external pressure or by some other long–ranged potential, the probability $\mathcal{P}_{3b}$ that all 3 strings form a local bound state behaves as $\mathcal{P}_{3b} \sim 1/\xi_{\|}{}^{\zeta_0(3)}$. As the line tension σ_2 of the interior string decreases, the angle θ decreases and the contact exponent ζ_0 increases. This is rather intuitive: as the interior string fluctuates more strongly, local contacts between the two outer strings become less likely.

The necklace model for m identical strings as described in Refs. [31, 32] also belongs to the intermediate fluctuation regime [33]. In this latter model, the strings again experience hard wall potentials between nearest neighbors and thus do not intersect whereas their attractive interaction is restricted to a short–ranged m–body potential. In this case, the parameter w has the value $w = [(m^2-3)^2 - 1]/4$ [33] and the two relations (4.18) and (4.19) for ζ_0 lead to $\zeta_0(m) = 1 \pm |m^2 - 3|/2$. For $m = 2$, one recovers the contact exponents for two strings and $w = 0$: there are no interior strings in this case and, therefore, there is no effective repulsion $\sim 1/l^2$.

For $m \geq 3$, on the other hand, the branch of $\zeta_0(m) = 1 \pm |m^2 - 3|/2$ with the minus sign corresponds to critical transitions in the presence of attractive m–body forces which again belong to subregime (C). Likewise, the branch with the plus sign again correponds to effectively repulsive m–body forces for which one has

$$\zeta_0(m) = (m^2 - 1)/2 \quad . \tag{4.23}$$

Now, this exponent governs the probability $\mathcal{P}_{mb}$ that all m identical strings form a local bound state, i.e., $\mathcal{P}_{mb} \sim 1/\xi_{\|}{}^{\zeta_0(m)}$ where a finite value of $\xi_{\|}$ is enforced by an external pressure or by another long–ranged potential.

The values (4.23) for the contact exponents have previously been derived by field-theoretic renormalization [11]; they are the scaling dimensions of the m-string contact operators (6.26) at the free Fermi fixed point (see (6.25) below). They govern the contact probabilities $\mathcal{P}_{mb}$ as long as all attractive interactions are sufficiently weak

In systems of several *nonintersecting* strings, there may be two-body and many-body interactions of either sign. Numerical transfer matrix results [8] as well as Monte Carlo simulations of bundles of strings or membranes [10] indicate in general a second order unbinding transition not in the universality class of the necklace model with a transition temperature that is independent of the number N of strings. This is understandable from the scaling picture [12] since with attractive pair forces the unbinding should be governed by $\mathcal{P}_{2b}$ alone. However, the effective critical exponents were found to depend on N over the numerically accessible range of scales [8, 10].

On the other hand, if only pair interactions are taken into account, the transfer matrix can be mapped onto that of a spin 1/2 xxz quantum spin chain; this model is soluble by a Bethe ansatz and yields the N-independent "Gaussian" exponents $\nu_\perp = 1$ and $\nu_\| = 2$ [9]. (Bethe ansatz methods may be extended to treat the unbinding of a system of such strings from a wall [13].)

The renormalization group discussed in Sect. 6.2 [11] reconciles these two results: if the unbinding is driven by pair forces, it is in the Gaussian universality class for an arbitrary number of strings, but the three-particle interactions contribute large corrections to scaling that may account for the N-dependence of the effective exponents. Moreover, there is a discrete sequence of new universality classes characterized by $\zeta_0 = 0$.

4.8 Extension to interfaces and membranes

The scaling picture for interacting strings as described above can be easily extended in the following way. First of all, the fluctuating humps of the manifolds will in general be governed by $\xi_\perp \sim {\xi_\|}^\zeta$ with $\zeta \neq 1/2$. As in the case of interacting strings, the roughness exponent ζ determines the fluctuation–induced interaction $\mathcal{V}_{fl}$ between the manifolds which can represent a loss of entropy or an increase in energy. The latter situation arises in systems with quenched or frozen randomness for which the manifolds are subject to a random potential. For thermally–excited fluctuations, one has

$$\mathcal{V}_{fl} \sim 1/\xi_\perp^\tau \qquad \text{with} \quad \tau = d_\parallel/\zeta \quad . \tag{4.24}$$

The case of 1–dimensional strings corresponds to $d_\parallel = 1$, $\zeta = 1/2$ and $\tau = 2$.

For fluctuations induced by quenched or frozen randomness, one has [34]

$$\mathcal{V}_{fl} \sim 1/\xi_\perp^\tau \qquad \text{with} \quad \tau = (2-2\zeta)/\zeta \quad . \tag{4.25}$$

Thus, the excess free energy of two interacting manifolds can now be written in the form

$$\Delta F \approx c_1 A/\xi_\perp^\tau + \mathcal{V}_{ub}(c_2\xi_\perp) + \Delta F_b/{\xi_\perp}^{\zeta_0/\zeta} \tag{4.26}$$

Minimization of this expression with respect to $\xi_\perp$ now leads to $\xi_\perp \sim 1/|\Delta F_b|^{\nu_\perp}$ with the roughening exponent

$$\nu_\perp = 1/(\tau - \zeta_0/\zeta) \quad . \tag{4.27}$$

One nontrivial check of this prediction can be obtained for wetting in 2–dimensional random bond systems. In this case, one has two interacting strings which feel a random potential with short–ranged correlations, and the roughness exponent has the value $\zeta = 2/3$ which implies the decay exponent $\tau = 1$. For a square–well potential, transfer matrix calculations using the replica trick lead to the scaling form $\mathcal{P}(l) \approx \Omega(l/\xi_\perp)/\xi_\perp$ for the probability distribution $\mathcal{P}(l)$ with the singular behavior $\Omega(s) \sim 1/s^{1/2}$ for small s, see [1], p.317. This implies $\zeta_0/\zeta = 1 - 1/2 = 1/2$. Thus, the excess free energy becomes

$$\Delta F \approx c_1 A/\xi_\perp + \Delta F_b/{\xi_\perp}^{1/2} \quad . \tag{4.28}$$

Minimization of this expression with respect to $\xi_\perp$ leads to the critical behavior

$$\xi_\perp \sim 1/|\Delta F_b|^{\nu_\perp} \qquad \text{with} \quad \nu_\perp = 2 \tag{4.29}$$

at the wetting transition. This agrees with the critical behavior as obtained via transfer matrix methods. [35, 36]

On the other hand, *repulsive* short–ranged potentials should also be characterized, in general, by a nontrivial value for the contact exponent ζ_0. The expression (4.26) for the excess free energy implies that, in this latter case, the exponent ζ_0 should satisfy the inequality $\zeta_0/\zeta \geq \tau$ which implies

$$\zeta_0 \geq d_\parallel \qquad \text{and} \quad \zeta_0 \geq 2(1-\zeta) \tag{4.30}$$

for thermally–excited fluctuations and for fluctuations excited by frozen randomness, respectively.

5 Field-theoretic renormalization I: An introductory example

In this section, we discuss in detail an ensemble of interacting random walks. This serves as an illustration of the field-theoretic methods† that are also important to the study of more general interfacial problems [7, 11]. Although the example is elementary it contains many aspects of renormalization group theory and can be read as a self-contained introduction to this subject. The results of the renormalization group calculation will be checked by independent means.

Random walks (or directed polymers) in $D = 1 + d'$ dimensions‡ are described by a d' -component position variable $\mathbf{l}(s)$, see fig. 3. (Interfaces in a two-dimensional system are the special case $d' = 1$.) Since these lines do not have overhangs, the free action for a single line is

$$\mathcal{H}_0(\mathbf{l}) = \frac{\sigma}{T} \int \left(\frac{\mathrm{d}\mathbf{l}(s)}{\mathrm{d}s} \right)^2 \mathrm{d}s \,. \tag{5.1}$$

We want to study the statistical effects of a short-ranged pair interaction between two such lines,

$$\mathcal{H} = \mathcal{H}_0(\mathbf{l}_1) + \mathcal{H}_0(\mathbf{l}_2) + \tilde{g} \int \delta(\mathbf{l}_2(s) - \mathbf{l}_1(s)) \mathrm{d}s \,. \tag{5.2}$$

The *scaling dimension* x_A of a variable A is defined by its transformation properties under scale transformations (i.e. substitutions $s \to s/b$ of the length variable): $A \to Ab^{x_A}$. The free action $\mathcal{H}_0$ contains two dimensionful variables, the position field of dimension -1 and the surface tension of dimension 1. The entire expression $\mathcal{H}_0$ is a pure number and hence of dimension 0. It is convenient to redefine the field variable, $\mathbf{z} = (\sigma/T)^{1/2}\mathbf{l}$ such that the free action in the new variable

$$\mathcal{H}_0(\mathbf{z}) = \int \left(\frac{\mathrm{d}\mathbf{z}(s)}{\mathrm{d}s} \right)^2 \mathrm{d}s \,, \tag{5.3}$$

†The approach described here, which is based on the critical operator algebra, is familiar in two-dimensional conformal field theory; see refs. [37, 38].

‡For notational simplicity, we call from now on d (before $d_\parallel$) the number of longitudinal and d' (before $d_\perp$) the number of transversal dimensions.

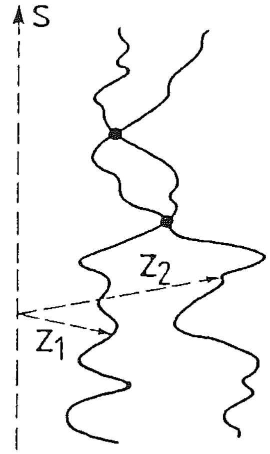

Fig. 3. Two directed lines in $D = 1+d'$ dimensions, described by d'-component canonical position variables $\mathbf{z}_i = (\sigma/T)^{1/2}\mathbf{l}_i$ $(i = 1, 2)$. The lines interact at their intersection points.

remains form invariant under scale transformations. This is called a *fixed point.* $\mathbf{z}$ has the *canonical* scaling dimension $-1/2$. The same substitution in the interaction defines the new interaction field

$$\Phi(s) \equiv \delta(\mathbf{z}_2(s) - \mathbf{z}_1(s)) \tag{5.4}$$

of canonical dimension

$$x = \frac{d'}{2} \tag{5.5}$$

and the conjugate coupling constant $g = (\sigma/T)^{-d'/2}\tilde{g}$ of dimension

$$y = 1 - x = \frac{2 - d'}{2} \,. \tag{5.6}$$

In the following assume that the transversal volume occupied by the two strings (given by the range of the vectors $\mathbf{z}_1$ and $\mathbf{z}_2$) is finite, for example that each component of these vectors is compactified to a circle of circumference $L_\perp \equiv L^{1/2}$. This *infrared regularization* introduces a finite probability that two *free* strings meet (and interact) at a given "time" s, namely

$$\langle \Phi(s) \rangle_0 \sim L^{-x} \,. \tag{5.7}$$

The full two-point function of the fields $\Phi(s_1)$ and $\Phi(s_2)$ measures the probability of two interactions taking place at times s_1 and s_2, i.e. the product of the probability for the first intersection (5.7) and their return probability to each other. At large distances $|s_2 - s_1| \gtrsim L$, this product approaches its asymptotic value $\langle \Phi(s_1) \rangle_0^2$. The connected correlation function has the form

$$\langle \Phi(s_1)\Phi(s_2) \rangle_0 = \langle \Phi(s_1) \rangle_0 \, |s_2 - s_1|^{-x} f(|s_2 - s_1|/L) \tag{5.8}$$

where $f(t) = 1$ for $t \ll 1$ and $f(t)$ is dominated by an exponential decay $\exp(-t)$ for $t \gtrsim 1$. The derivation of this simple consequence of (5.3) is left to the reader. The relation

$$\Phi(s_1)\Phi(s_2) = |s_2 - s_1|^{-x} \Phi(s_1) + \ldots \tag{5.9}$$

for $|s_2 - s_1| \ll L$ is valid even as an operator identity, i.e. when inserted into an arbitrary correlation function $\langle \ldots \Phi(s_1)\Phi(s_2) \ldots \rangle_0$. (The less singular terms omitted on the r.h.s. contain gradient fields.) This is the simplest example of an operator algebra describing universal short-distance properties of the correlation functions. It is important to note that the operator algebra is independent of the infrared regularization.

The scale L serves not only to regularize the correlation functions but also as a macroscopic unit of length for the system. We use it to define the *dimensionless coupling constant*

$$u = g L^y \tag{5.10}$$

and the dimensionless free energy of a system of transversal length $L_\perp \equiv L^{1/2}$ and longitudinal length $L_\parallel$ in the thermodynamic limit of $L_\parallel$:

$$F = - \lim_{L_\parallel \to \infty} \frac{L}{L_\parallel} \int \mathcal{D}\mathbf{z} \exp(-\mathcal{H}(\mathbf{z}; L_\parallel, L)) \,. \tag{5.11}$$

L is not an intrinsic scale of the system, but an external experimental variable. The response of the coupling constant u to a change of L is called its *beta function*:
$L\partial_L u \equiv \beta(u)$. From (5.10), we obtain

$$\beta(u) = y u \,. \tag{5.12}$$

This flow equation is depicted in fig. 4 (a). It has two *fixed points* (i.e. values of u that remain invariant under a change in L): the free action fixed

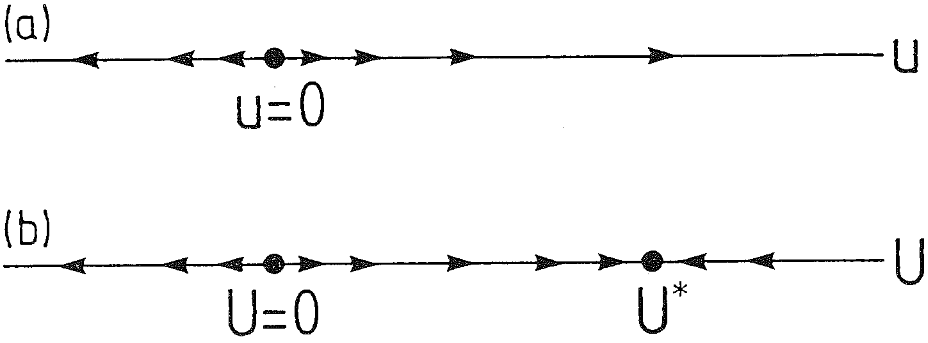

Fig. 4. Renormalization group flow. (a) In the unrenormalized coupling constant u, the two fixed points are $u = 0$ and $U = U^\star$. (b) In the renormalized coupling U, the fixed points are $U = 0$ and $U = U^\star$.

point $u = 0$ and the fixed point $u = \infty$. With increasing parameter L, all values of u flow from an *unstable* fixed point to a *stable* fixed point that hence characterizes the behavior on large scales. For $y < 0$ (i.e. $d' > 2$), the free action is stable (the interaction $g\Phi$ is then called an *irrelevant* perturbation of the free theory); for $y > 0$ (i.e. $d' < 2$), it is unstable (the interaction is then called a *relevant* perturbation), and the long-distance behavior is governed by the fixed point $u = \infty$.

We want to calculate this long-distance behavior of the interacting theory in terms of the correlation functions (5.7) and (5.8) of the free theory. We can write the free energy (5.11) as a power series in the dimensionless coupling constant,

$$F(u) - F(0) = \sum_{N=1}^{\infty} \frac{(-1)^N}{N!} u^N L^{1-Ny} I_N \,, \tag{5.13}$$

where

$$I_N = \int \langle \Phi(s_1)\Phi(s_2)\ldots\Phi(s_N)\rangle_0 \mathrm{d}s_2 \ldots \mathrm{d}s_N \tag{5.14}$$

involves connected correlation functions in the unperturbed theory. But it is easy to see that for a generic field theory, this power series can be formal at best. The scaling dimension of the integral I_N is $1 - Ny$. Hence for $y > 0$, each individual term $I_N \sim L^{Ny-1}$ of order $N \geq 1/y$ diverges for $L \to \infty$, although the sum (5.13) is of course finite. This *infrared divergence* is the reason why the perturbation series has to be renormalized. For $y < 0$, already

I_2 is *ultraviolet divergent*, i.e. the integral

$$\iota(y) = \int_0^\infty |s_2 - s_1|^{-1+y} f(|s_2 - s_1|/L) \mathrm{d}s_2 \qquad (5.15)$$

does not exist because of the singularity of the integrand (5.8) as $s_2 \to s_1$. This singularity appears as a pole

$$\iota(y) = \frac{1}{y} + O(y^0) \qquad (5.16)$$

for $y \searrow 0$. The crucial property of renormalizable theories is that this difficulty can be overcome by a *nonlinear* change of variables. To this end, we insert (5.7) and (5.8) into (5.13),

$$F(u) - F(0) = -\langle\Phi\rangle_0 \, L^x[u - \iota\, u^2 + O(u^3)] \,, \qquad (5.17)$$

and call the expression in square brackets on the r.h.s. the *renormalized coupling constant u*:

$$U = u - \iota\, u^2 + O(u^3) \,. \qquad (5.18)$$

Then it is obvious that F as a function of U does not have any singularities; we simply have

$$F(U) - F(0) = -\langle\Phi\rangle_0 \, L^x U \,, \qquad (5.19)$$

where $\langle\Phi\rangle_0 \, L^x$ is a constant of the free theory by (5.7) and hence independent of U. Instead, the transformation of variables (5.18) becomes singular as $y \to 0$. At first sight, this just looks like a dirty trick. In order to see what really happens, we carry out the change of variables (5.18) in the flow equation (5.12):

$$\beta(U) = \frac{\mathrm{d}U}{\mathrm{d}u}\beta(u) = yU - \iota\, yU^2 + O(U^3) \,. \qquad (5.20)$$

For the particular system of this example, this equation is even valid exactly, i.e. the power series in U terminates at second order §.

§This is because (5.13) is a geometric series,

$$F(u) - F(0) = -\langle\Phi\rangle \, L^x \frac{u}{1 + \iota u} \,,$$

and hence $U = u/(1 + \iota u)$ exactly. In other regularization schemes, the perturbation series is summable in a similar way [39].

As shown in fig. 4 (b), the stable fixed point $u = \infty$ for $y > 0$ appears at a finite value

$$U^\star = \frac{1}{\iota} \tag{5.21}$$

of the renormalized variable U. This is only possible because the transformation of variables (5.18) is nonlinear. For small values of y, the fixed point $U^\star$ approaches the free fixed point $U = 0$; the two fixed points coalesce for $y = 0$.

Figs. 4 (a) and (b) describe the same flow field on the space of interacting theories, albeit in different "coordinate systems". Hence the singularities of the free energy in the unrenormalized variable u can be understood as coordinate singularities [38]. In general relativity, coordinate singularities are well known; the most famous example is Schwarzschild coordinates at the event horizon of a Schwarzschild black hole. They can be removed by choosing a suitable coordinate system. *Renormalization is nothing but the choice of a suitable coordinate system on the space of interactions.*

The large-scale behavior for $y > 0$ can be read off more easily from the renormalized theory. For attractive coupling ($U < 0$), the flow of fig. 4 (b) does not have a fixed point and the absolute value of the coupling constant grows indefinitely as $L \to \infty$: the system has a finite correlation length. It is in a bound state, the average transversal distance of the two strings remains finite. For $d' = 1$, we can check this by looking at the quantum-mechanical Hamiltonian

$$\hat{H} = -\frac{\partial^2}{\partial z_1^2} - \frac{\partial^2}{\partial z_2^2} + g\delta(z_2 - z_1) \tag{5.22}$$

resulting from the action (5.2): the ground state in the relative coordinate $z_1 - z_2$ is always a bound state for $g < 0$.

For repulsive coupling ($U > 0$), the behavior on large scales is governed by the fixed point $U^\star$. At this fixed point, the field Φ acquires a new scaling dimension $x^\star$ determining the power law decay of its correlation functions (5.7), (5.8) etc. To show this, we express the flow of the free energy in both renormalized and unrenormalized couplings,

$$L\partial_L F = \frac{\mathrm{d}F}{\mathrm{d}U}\beta(U) = \frac{\mathrm{d}F}{\mathrm{d}u}\beta(u)\,, \tag{5.23}$$

where U and u are linked by eq. (5.18). We obtain the asymptotic behavior for large L

$$\beta(u(L)) = yu(L) \sim L^y \tag{5.24}$$

and

$$\beta(U(L)) \simeq y^\star(U(L) - U^\star) \sim L^{y^\star} \tag{5.25}$$

with $y^\star \equiv (\mathrm{d}/\mathrm{d}U)\beta(U)|_{U=U^\star} = -y = (d' - 2)/2$ from the linearized flow equation. (In a generic field theory, there are terms of cubic and higher order in the beta function; hence $y^\star = -y + O(y^2)$.) Since $\mathrm{d}F/\mathrm{d}U$ is a constant by (5.19), we have

$$\frac{\mathrm{d}F}{\mathrm{d}u} \sim L^{y^\star - y} \tag{5.26}$$

and

$$\langle \Phi \rangle \sim \frac{1}{L}\frac{\mathrm{d}F}{\mathrm{d}g} \sim L^{-x^\star} \tag{5.27}$$

with

$$x^\star = 1 - y^\star \, . \tag{5.28}$$

Hence the correlation functions of the field Φ decay faster than for a free system since the repulsive interaction effectively suppresses intersections of the two lines. For $d' = 1$, if intersections are completely suppressed, the two lines can be regarded as the world lines of two particles obeying the Pauli exclusion principle [40]. Thus the fixed point $U^\star$ describes a system of *free fermions*. It is easy to verify that the exponents $y^\star = -1/2, x^\star = 3/2$ are indeed characteristic of free fermions. Consider the fermionic analogon of the operator Φ, namely $\bar{\Phi} \equiv \delta(\phi_2 - \phi_1 - a)$, where a is some fixed microscopic transversal distance. The expectation value of $\bar{\Phi}(s)$ in a two-particle state $|\chi\rangle$,

$$\langle \chi | \bar{\Phi}(s) | \chi \rangle = \int |\chi(z_1, z_1 + a)|^2 \mathrm{d}z \tag{5.29}$$

has the scaling form $L^{-1/2}|f(aL^{-1/2})|^2$, where $f(\varepsilon) \sim \varepsilon$ due to the antisymmetry of the fermionic wave function.

6 Field-theoretic renormalization II: More general interfacial problems

In the previous section, we have examined the effect of a short-ranged interaction on a system of two directed strings in $D = 1 + d'$ dimensions using methods of renormalized continuum field theory. We have found a new universality class that describes the large-scale behavior for $d' < 2$ and can be interpreted as an effective low-energy Fermi theory for $d' = 1$. This is the reason why quite a few problems in $D = 1+1$, e.g. the statistics of steps on crystal surfaces, may be formulated in terms of (interacting) fermions.

In this section we generalize this approach in order to treat a variety of interfacial problems most of which are not so readily accessible by other means [7, 11, 14]. The strategy is always to identify the continuum fields relevant to the problem, to write down their operator algebra and hence to infer the renormalization group equations to leading order.

6.1 Long-range interactions

Consider again a system of two directed lines in $D = 1+1$ dimensions. An obvious way to generalize the interaction discussed in the previous section is to allow also for long-ranged forces in the transversal direction. This is described by the action

$$\mathcal{H} = \mathcal{H}_0(z_1) + \mathcal{H}_0(z_2) + \int [(g - g_c)\,\Phi + h\,\Omega_\rho] \mathrm{d}s \tag{6.1}$$

where

$$\Phi(s) = \sqrt{\pi}\delta(z(s)) \tag{6.2}$$

and

$$\Omega_\rho(s) = \frac{1}{4}|z(s)|^{-\rho} \tag{6.3}$$

describe the short-ranged part and the long-ranged tail of the interaction potential. (The numerical factors are only a matter of convenience.) The distinction between these two parts is not unique since it involves some microscopic scale $a_\perp$. Therefore the field Φ has a nonuniversal critical strength g_c that depends on $a_\perp$. The field-theoretic renormalization for this system follows [14]; it is carried out using the approach of [7].

The fields $\Phi(s)$ and $\Omega_\rho(s)$ have scaling dimensions

$$x = \frac{1}{2} \tag{6.4}$$

and

$$x_\rho = \frac{\rho}{2}\,, \tag{6.5}$$

respectively. The conjugate coupling constants have dimensions $y = 1/2$ and $y_\rho = 1 - x_\rho$, we define the dimensionless couplings $u = gL^y$ and $v = hL^{y_\rho}$. The extended operator algebra is

$$\Phi(s_1)\Phi(s_2) = C^{\Phi}_{\Phi\Phi}|s_2 - s_1|^{-x}\,\Phi(s_1) + \dots \tag{6.6}$$

$$\Phi(s_1)\Omega_\rho(s_2) = C^{\Phi}_{\Phi\Omega}|s_2 - s_1|^{-x_\rho}\,\Phi(s_1) + \dots \tag{6.7}$$

$$\Omega_\rho(s_1)\Omega_\rho(s_2) = C^{\Phi}_{\Omega\Omega}|s_2 - s_1|^{-2x_\rho + x}\,\Phi(s_1) + C^{\Omega^2}_{\Omega\Omega}\,\Omega_{2\rho}(s_1) + \dots \tag{6.8}$$

with $C^{\Phi}_{\Phi\Phi} = 1, C^{\Phi}_{\Phi\Omega} = 2^{-\rho}\pi^{-1/2}\Gamma((1-\rho)/2), C^{\Phi}_{\Omega\Omega} \neq 0$ and $C^{\Omega^2}_{\Omega\Omega} = 1/4$. The structure constants not explicitly written are zero, in particular $C^{\Omega}_{\Phi\Omega}$ and $C^{\Omega}_{\Omega\Omega}$. The resulting renormalization group equations for the renormalized couplings U and V $(= v)$ are

$$L\partial_L U \equiv \beta_U(U,V) = (y - 2C^{\Phi}_{\Phi\Omega}V + O(V^2))U - C^{\Phi}_{\Phi\Phi}U^2\,, \tag{6.9}$$

$$L\partial_L V \equiv \beta_V(U,V) = y_\rho V\,. \tag{6.10}$$

For $\rho \neq 2$, the critical roughening transition is always determined by the Gaussian fixed point $U = V = 0$. For a generic perturbation of the form $g\,\Phi + h\,\Omega_\rho$, e.g. the critical exponent $\nu_\parallel$ of the correlation length $\xi_\parallel$ is determined by the more relevant of the two fields, i.e. $\nu_\parallel = 1/y_\rho = 2/(2-\rho)$ for $\rho \leq 1$ and $\nu_\parallel = 1/y = 2$ for $\rho \geq 1$.

The case $\rho = 2$ is more involved. The flow of U and V is shown in fig. 5. The field Ω_2 is marginal at the Gaussian fixed point, and the structure of the operator algebra implies that it remains marginal perturbatively in U and V to all orders: $\beta_V(U,V) = 0$. Hence the Gaussian fixed point is part of the fixed line $U = 0$, which is parametrized by V. Along this line, the scaling dimension of Φ varies according to

$$x(V) = 1 - y(V) = x + 2C^{\Phi}_{\Phi\Omega}V + O(V^2) \tag{6.11}$$

with $C^{\Phi}_{\Phi\Omega} = -1/2$. Perturbation with the relevant coupling $U > 0$ generates, just as in the previous section, a crossover to the stable fixed point $U^\star(V) = y(V)/C^{\Phi}_{\Phi\Phi}$ (the beta function $\beta_U(U,V)$ is again strictly quadratic in U). Hence the free Fermi fixed point $U^\star(0)$ is part of a fixed line as well. This line is the fermionic analogon of the fixed line $U = 0$; it describes *nonintersecting lines with* $(1/z^2)$*-interactions.* The scaling dimension of the irrelevant field $\bar{\Phi}$ varies according to

$$x^\star(V) = 1 - y^\star(V) = 1 + y(V)\,. \tag{6.12}$$

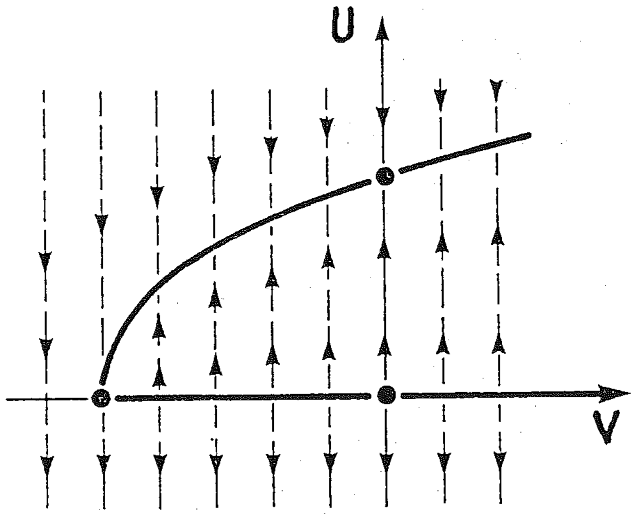

Fig. 5. Renormalization group flow for the couplings U and V conjugate to the short-ranged part and the long-ranged part of the potential, respectively, in the case $\rho = 2$. The coupling V remains strictly marginal. It generates the line of fixed points $U = 0$ describing interacting bosonic lines and the line $U = U^{\star}(V)$ describing interacting fermions.

One can show [27] that for $V < -1/4$, the system is always in a bound state. Hence at that value of V, the two lines of fixed points have to join, an example of a parabolic renormalization group flow [26].

6.2 A system of many lines

Consider a system of many directed strings in $D = 1 + d'$ dimensions. We follow the discussion of [11]. In real systems, the interaction between particles is certainly more complicated than the simple pair force considered above. Typically, the force between two lines is screened or enhanced by the presence of further lines. Casimir-type many-body forces (which may be screened at some distance) arise from the coupling of the strings to the surrounding medium, e.g. a correlated fluid [41]. It is convenient to use a description in second quantization, which is valid for an arbitrary number of lines. If we restrict ourselves to short-ranged interactions, such a system may be described by the continuum Hamiltonian

$$\hat{H} = \int (\partial_{\mathbf{r}}\phi^{\dagger}(\mathbf{r},s))(\partial_{\mathbf{r}}\phi(\mathbf{r},s))\mathrm{d}^{d'}\mathbf{r} + \sum_{m\geq 2} g_m \Phi_m(s) \,. \tag{6.13}$$

which is independent of the number of strings. (This number only enters the in- and out-states and hence does not affect the renormalization.) The Bose creation and annihilation operators ϕ and $\phi^\dagger$ obey canonical commutation relations and have dimension $d'/4$. The normal-ordered vertex

$$\Phi_m(s) = \frac{1}{m!}\int (\phi^\dagger(\mathbf{r},s))^m(\phi(\mathbf{r},s))^m \mathrm{d}^{d'}\mathbf{r} \tag{6.14}$$

describes the interaction of m strings at a common intersection point s. (The lowest vertex Φ_2 has been denoted by Φ in the last section.) In the space of these interactions, we will find a whole sequence of new universality classes describing the roughening of a finite number of strings.

The vertices Φ_m and their conjugate coupling constants g_m have scaling dimensions

$$x_m = (m-1)\frac{d'}{2} \tag{6.15}$$

and $y_m = 1 - x_m$, respectively. One shows that the vertices form a short-distance algebra of the form

$$\Phi_k(s_1)\Phi_l(s_2) = \sum_{m=\max(k,l)}^{k+l-1} C_{kl}^m \, |s_2 - s_1|^{-(k+l-m-1)d'/2}\, \Phi_m(0) + \ldots; \tag{6.16}$$

each term corresponds to a real-space Feynman diagram with $k+l-m$ lines joining the two vertices. The combinatorial factors are (normalized such that $C_{22}^2 = 1$)

$$C_{kl}^m = \frac{m!}{(m-k)!(m-l)!(k+l-m)!}\left(\frac{k+l-m}{2}\right)^{-d/2}. \tag{6.17}$$

The correlation functions of the vertices Φ_m are again defined by means of an infrared regularization; its scale parameter L is used to define the dimensionless coupling constants $u_m = g_m L^{y_m}$ and governs the renormalization group flow. The regularization involves analytic continuation in d' and consists in absorbing the singularities in the perturbation expansion for $F(u_2, u_3, \ldots)$ into renormalized couplings U_m. The fixed points of the flow equations $L\partial_L U_m \equiv \beta_m(U_2, U_3, \ldots)$ determine the universality classes.

First, the short-distance singularities of the series $F(u_2, 0, \ldots)$ at $y_2 = 0$ (i.e. $d' = 2$) are determined by the operator product

$$\Phi_2(s_1)\Phi_2(s_2) = |s_2 - s_1|^{-x_2}\, \Phi_2(s_1) + \ldots \tag{6.18}$$

and lead to the beta function

$$\beta_2(U_2) = y_2 U_2 - U_2^2 \tag{6.19}$$

for the renormalized pair coupling (5.18), as discussed in the previous section. This coupling constant renormalization makes F finite as a function of U_2, but singularities still exist in the dependence on the higher couplings $u_m (3 \leq m)$. The singularities linear in u_m correspond to singularities of the correlation functions $\langle \ldots \Phi_m(s) \ldots \rangle$ as a function of u_2 only. By inserting the operator product

$$\Phi_m(s_1)\Phi_2(s_2) = C_{m2}^m |s_2 - s_1|^{-x_2} \, \Phi_m(s_1) + \ldots \tag{6.20}$$

into the perturbation series, we obtain

$$F(u_2, u_m) = F(u_2, 0) - \langle \Phi_m \rangle L^{x_m} [u_m - 2C_{m2}^m \, \iota(y) u_m u_2 + O(u_m u_2^2, u_m u_k] \tag{6.21}$$

where $3 \leq k \leq m$ and $\iota(y)$ denotes the integral (5.15). We remove these singularities by defining

$$U_m = u_m - 2C_{m2}^m \, \iota \, u_m u_2 + O(u_m u_2^2, u_m u_k) \tag{6.22}$$

which leads to the beta functions

$$\beta_m(U_2, \ldots, U_m) = y_m(U_2) U_m + O(U_k U_m) \tag{6.23}$$

with

$$y_m(U_2) = y_m - 2C_{m2}^m U_2 + O(U_2^2) \, . \tag{6.24}$$

For $m \geq 3$, (6.24) does not terminate at first order. In $d' = 1$, however, the contribution from higher orders miraculously vanishes at the fermionic fixed point $U_2^\star = 1/2$, so that the infrared scaling dimensions resulting from (6.24), (6.17), and (6.19),

$$\bar{x}_m = 1 - y_m(U_2^\star) = \frac{m^2 - 1}{2} \, , \tag{6.25}$$

are exact. It is easy to show that they are precisely the scaling dimensions of the fermionic multi-particle operators

$$\bar{\Phi}_m(t) \equiv \frac{1}{m!} \int \prod_{i=1}^{m} \psi^\dagger(r + a_i) \psi(r + a_i) \mathrm{d}r \, , \tag{6.26}$$

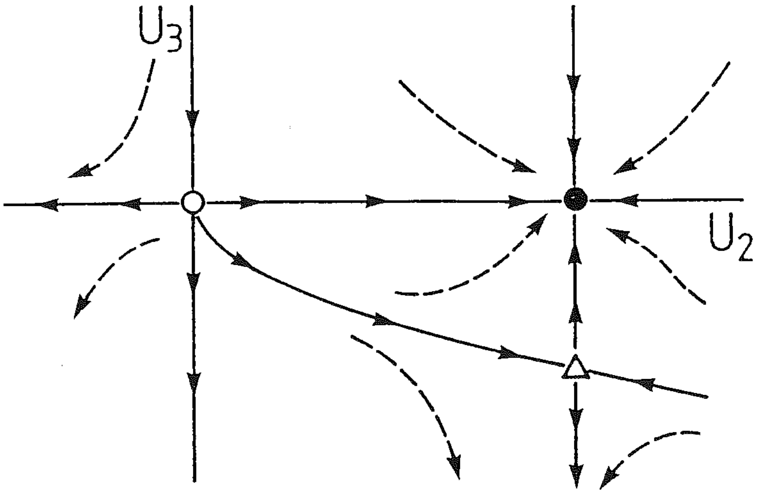

Fig. 6. The renormalization group flow in the space of two- and three-body coupling constants U_2, U_3 has three fixed points. The free Bose fixed point ($\circ$) describes the roughening transition driven by two-particle interactions, the free Fermi fixed point ($\bullet$) the unbound phase of nonintersecting lines, and the interacting Fermi fixed point ($\triangle$) the roughening of fermionic lines driven by three-particle interactions.

where a_i are fixed microscopic distances ¶.

The full beta function for U_3 now follows in a similar way from the singularities in the series $F(u_2, u_3)$ at $y_3 = 0$, which are determined by the operator product

$$\Phi_3(s_1)\Phi_3(s_2) = C_3|s_1 - s_2|^{-x_3}\,\Phi_3(s_1) + \ldots \tag{6.27}$$

(with $C_3 \equiv C_{33}^3$). We obtain

$$\beta_3(U_2, U_3) = (y_3 - 3U_2)U_3 - C_3U_3^2\,. \tag{6.28}$$

In general, it follows from (6.16) that the beta function $\beta_m(U_2, U_3, \ldots)$ depends only on the U_k with $k \leq m$.

Fig. 6 shows the renormalization group flow in the space of two- and three-string couplings given by (6.19) and (6.28) for $d' = 1$. It has three fixed points: the Gaussian fixed point $U_2 = U_3 = 0$ describing free bosonic lines, the free Fermi fixed point $U_2 = y_2, U_3 = 0$ discussed in the previous section, and the new fixed point $U_2 = y_2, U_3 = \bar{y}_3/C_3$, a theory of interacting fermions. The free Fermi fixed point is completely stable; all short-ranged

¶These operators were first discussed in [31, 32] without the explicit use of free fermions.

interactions are irrelevant at this fixed point. Hence *weak* forces do not alter the asymptotic behavior at large distances. The new fixed point is an unstable fixed point. It describes a critical roughening transition at *finite* three-particle interaction strengh that marks the transition to a fermionic bound state. From the foregoing discussion it is clear that this fixed point is just the first member of a whole family of fermionic universality classes represented by fixed points of the higher multi-particle interactions Φ_m. Thus the interplay of attractive and repulsive forces generates a rich scenario of universality classes of interacting directed lines. We emphasize again that this scenario applies to an arbitrary finite number of lines.

6.3 Interfaces in general dimensionality

In this section, we study interfaces in D dimensions, i.e. directed manifolds of dimension d that are embedded in a space of dimension $D = d + 1$ and are described by single displacement field $z(\mathrm{s})$. We are again interested in a pair of such manifolds coupled by a short-ranged interaction

$$\mathcal{H}(z_1, z_2) = \mathcal{H}_0(z_1) + \mathcal{H}_0(z_2) + \int \mathcal{V}(z_2(\mathrm{s}) - z_1(\mathrm{s}))\mathrm{d}^d\mathrm{s}\,. \tag{6.29}$$

But now we allow for an internal structure of the manifolds that results in a more complex functional form of the potential $\mathcal{V}(z)$ with possibly more than one extremum. In the space of these general short-ranged potentials, we will again find a whole family of fixed points that represent universality classes of critical roughening. We follow the treatment of [7].

In the continuum limit, where the microscopic range a of the potential approaches zero, we expand a generic potential in the basis of scaling fields

$$\Phi_\alpha(\mathrm{s}) \equiv \sqrt{4\pi}\left(\frac{\partial}{\partial z(\mathrm{s})}\right)^\alpha \delta(z(\mathrm{s})) \qquad (\alpha = 0, 1, 2, \ldots)\,. \tag{6.30}$$

(In particular, potentials with a single extremum are described by the field Φ_0, which has been called Φ in sect. 5.)

These Gaussian fields have canonical scaling dimensions

$$x_\alpha = (\alpha + 1)\zeta\,, \tag{6.31}$$

their conjugate coupling constants have dimensions

$$y_\alpha = d - x_\alpha = \frac{\alpha + 3}{2}\left(d - 2 + \frac{4}{\alpha + 3}\right)\,. \tag{6.32}$$

In any dimension $d < 3$, only finitely many of these fields are relevant; they span the space of bona fide renormalizable binding potentials. The fields $\Phi_0, \Phi_2 \ldots$ are even, the fields $\Phi_1, \Phi_3, \ldots$ are odd under the $\mathbb{Z}_2$-symmetry $z \to -z$ of the Gaussian theory.

In order to define correlation functions of the Φ_α, we again compactify the displacement field to a circle of circumference $L_\perp = L^\zeta$. (Alternatively, we could add a "mass term" $\int \mu^2 z^2 \mathrm{d}^d\mathrm{s}$ to the action (6.29).) ¿From the short-distance structure of the correlation functions we extract the operator algebra

$$\Phi_\alpha(\mathrm{s}_1)\Phi_\beta(\mathrm{s}_2) = \sum_\gamma C^\gamma_{\alpha\beta}\,|\mathrm{s}_2 - \mathrm{s}_1|^{-x_\alpha - x_\beta + x_\gamma}\Phi_\gamma(\mathrm{s}_1) + \ldots\,, \tag{6.33}$$

with coefficients

$$C^\gamma_{\alpha\beta} = \sum_{\delta_1=0}^{\alpha}\sum_{\delta_2=0}^{\beta}(-1)^{\alpha-\delta_2}(-2)^{-\frac{1}{2}(\alpha+\beta+\gamma)}\binom{\alpha}{\delta_1}\binom{\beta}{\delta_2}\frac{c(\alpha+\beta-\delta_1-\delta_2)}{(\gamma-\delta_1-\delta_2)!!} \tag{6.34}$$

if $\alpha + \beta + \gamma$ is even; otherwise they vanish by symmetry. $c(\delta)$ is defined as $(\delta - 1)!!$ if δ is even and 0 otherwise, and only terms with $\delta_1 + \delta_2 \leq m$ are included in the sum.

Consider the perturbation series $F(u_\gamma)$ for a given symmetric potential Φ_γ $(\gamma = 0, 2, \ldots)$. The coupling constant g_γ is relevant if d is *larger* than the borderline dimension $d_\gamma = 2 - 4/(\gamma + 3)$. We now use d instead of d' as analytic parameter to regularize this series. The ultraviolet singularities of I_2 are determined by the operator product $\Phi_\gamma(\mathrm{s}_1)\Phi_\gamma(\mathrm{s}_2)$, additional singularities appear at higher order. For small positive y_γ, the terms

$$\Phi_\gamma(\mathrm{s}_1)\Phi_\gamma(\mathrm{s}_2) = \ldots + C^\alpha_{\gamma\gamma}|\mathrm{s}_2 - \mathrm{s}_1|^{-2x_\gamma + x_\alpha}\,\Phi_\alpha(\mathrm{s}_1) + \ldots \tag{6.35}$$

with $\alpha < \gamma$ generate power-like singularities proportional to $a^{(\alpha-\gamma)\zeta + y_\gamma}$ if the integral is defined with a short-distance cutoff a. They only lead to a cutoff-dependent shift in the critical values of g_γ and are automatically subtracted if the integral is defined by analytic continuation to higher dimensions. The term

$$\Phi_\gamma(\mathrm{s}_1)\Phi_\gamma(\mathrm{s}_2) = \ldots + C^\gamma_{\gamma\gamma}|\mathrm{s}_2 - \mathrm{s}_1|^{-x_\gamma}\Phi_\gamma(\mathrm{s}_1) + \ldots \tag{6.36}$$

generates a pole in y_γ which is to be absorbed into the definition of the renormalized coupling U_γ. Hence we obtain the beta function

$$L\partial_L U_\gamma \equiv \beta_\gamma(U_\gamma) = y_\gamma U_\gamma - \frac{S_\gamma}{2}C^\gamma_{\gamma\gamma}\,U_\gamma^2 + O(U_\gamma^3)\,, \tag{6.37}$$

where s_γ denotes the surface of the d_γ-dimensional unit sphere. It has the infrared fixed point $U_\gamma^\star = (2/s_\gamma C_{\gamma\gamma}^\gamma) y_\gamma + O(y_\gamma^2)$.

Additional singularities appear in the perturbation expansion of the correlation functions as a function of U_γ; they determine the new infrared scaling dimensions

$$x_\alpha^\star = x_\alpha + \frac{2C_{\alpha\gamma}^\alpha}{C_{\gamma\gamma}^\gamma} y_\gamma + O(y_\gamma^2)\,. \tag{6.38}$$

Thus for each $\gamma = 0, 2, \ldots$, we obtain an interacting continuum field theory $\mathcal{T}_\gamma$ that describes universal long-distance behavior with a potential Φ_γ at the critical roughening point above the borderline dimension d_γ; below that dimension, the fixed point $\mathcal{T}_\gamma$ is unstable and, at least for a sufficiently weak potential strength, the long-distance behavior of the system is Gaussian. The theory $\mathcal{T}_0$ governs the scaling of an unbound interface subject to a purely repulsive potential, it is the dimensional continuation in d of the free Fermi fixed point in $D = 1 + 1$ discussed above. It has been shown rigorously that this theory is renormalizable to all orders [42]. The higher theories $\mathcal{T}_2, \mathcal{T}_4, \ldots$ form a hierarchy of multicritical universality classes: the fixed point $\mathcal{T}_\gamma$ has the γ relevant scaling fields $\Phi_0, \Phi_1, \ldots, \Phi_{\gamma-1}$ (the field $\Phi_{\gamma+1}$ is redundant and the fields $\Phi_\gamma, \Phi_{\gamma+2}, \Phi_{\gamma+3}, \ldots$ are irrelevant), and we expect a series of cross-over phenomena $\mathcal{T}_\gamma \to \mathcal{T}_{\gamma-2} \to \ldots \to \mathcal{T}_0$.

This hierarchy of universality classes mirrors in a remarkable way the well-known series of bulk multicritical points in Ising sytems. The latter series is represented by actions

$$\mathcal{H} = \int [(\nabla\phi)^2 + g_\gamma \phi^\gamma] \mathrm{d}^D \mathbf{r} \qquad (\gamma = 4, 6, \ldots) \tag{6.39}$$

in terms of the local order parameter $\phi(\mathbf{r})$. Nontrivial renormalization group fixed points bifurcate from the Gaussian fixed point at borderline dimensions $D_\gamma = 2 + 4/(\gamma - 2)$ and describe fluctuation-dominated critical ($\gamma = 4$), tricritical ($\gamma = 6$) or higher multicritical behavior *below* that dimension (see fig. 7). In $D = 2$, infinitely many such fixed points exist; they form the well-known series of minimal conformal field theories [43]. The status of the universality classes $\mathcal{T}_\gamma$ in $d = 2$ remains a challenging open question. Are they related to conformal field theories as well?

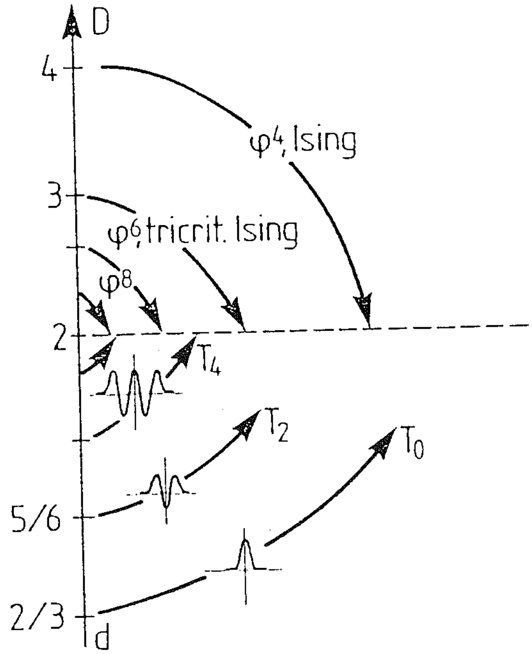

Fig. 7. Fluctuation-dominated universality classes bifurcating from the Gaussian theory. The series of Ginzburg-Landau theories with monomial interactions ϕ^γ ($\gamma = 4, 6, \ldots$) has borderline dimensions $D_\gamma = 2 + 4/(\gamma + 2)$. These theories are well-defined in $D = 2$, where they are the series of minimal conformal field theories. The series of roughening transitions $\mathcal{T}_\gamma$ ($\gamma = 1, 2, \ldots$) has borderline dimensions $d_\gamma = 2 - 4/(\gamma + 3)$.

6.4 Outlook

As the discussion in this section shows, interface criticality can in many cases be understood on an equal footing with bulk criticality, namely as a renormalized continuum field theory (living in the dimensionality $d = D-1$ of the interface) whose correlation functions satisfy a well-defined short-distance algebra.

However, it is as yet difficult to incorporate transitions of ($d = 2$)-dimensional interfaces (which are only logarithmically rough without interactions) into this framework. In this dimension, conformal field theory should come into play. Further questions arise from the breaking of the $\mathbb{Z}_2$-symmetry e.g. for wetting transitions.

A related case of interest is that of membranes, i.e. manifolds governed by bending rigidity instead of surface tension. Even if we consider membranes below the persistence length (so that a gradient approximation is valid) the difference in the kinetic part of the action is expected to introduce modifications to the scenario described here.

Perhaps most importantly, criticality of directed lines in random media (which is related to criticality of nonequilibrium growth models) can be formulated in replica language in much the same way as the problems discussed here. Whether it can be understood as a proper renormalized field theory remains to be seen.

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