EUROPHYSICS LETTERS 10 October 1994

Europhys. Lett., 28 (2), pp. 103-108 (1994)

Bundles of Interacting Strings in Two Dimensions.

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(received 17 June 1994; accepted in final form 8 September 1994)

PACS. 68.10 – Fluid surfaces and interfaces with fluids (inc. surface tension, capillarity, wetting and related phenomena).
PACS. 64.70 – Phase equilibria, phase transitions, and critical points.
PACS. 82.70 – Disperse systems.

Abstract. – Bundles of strings which interact via short-ranged pair potentials are studied in two dimensions. The corresponding transfer matrix problem is solved analytically for arbitrary string number N by Bethe ansatz methods. Bundles consisting of N identical strings exhibit a unique unbinding transition. If the string bundle interacts with a hard wall, the bundle may unbind from the wall via a unique transition or a sequence of N successive transitions. In all cases, the critical exponents are independent of N and the density profile of the strings exhibits a scaling form that approaches a mean-field profile in the limit of large N.

In the context of condensed-matter physics, strings are essentially 1-dimensional objects which are i) directed, in the sense that their tangent vectors point, on average, into a certain direction, and ii) are governed by a finite line tension. Physical examples are domain walls in adsorbed monolayers [1], steps or ledges on crystal surfaces [2], vortex lines in type-II superconductors [3], stretched polymers [4] and presumably some polyelectrolytes [5]. Two different ensembles of strings have to be distinguished: i) systems with a fixed density of strings and ii) systems with a fixed number N of strings, which are the topic of this letter. If the strings have attractive interactions, they may at low temperatures be bound together to a bundle. Such bundles of strings have been studied by numerical diagonalization of the transfer matrix [6, 7], in a local density functional theory [8], by mapping onto a quantum spin chain [9], in a heuristic scaling picture [10], and by field-theoretic renormalization group methods [11].

In this paper, we study bundles of N strings which interact via contact pair potentials. Such a system of strings can be mapped onto a system of N quantum-mechanical particles interacting via the same pair potentials. For N strings in two dimensions, one is then led to consider a Schrödinger-type equation in N dimensions which can be solved analytically for *arbitrary* N using the Bethe product ansatz (see *e.g.* [12] for a review). Often, real strings do not intersect which can be taken into account by imposing Fermi statistics on the particles [4, 1]. Here, we take a different avenue. We modify the contact interaction so as to

impose a *preferred ordering* on the strings while *preserving integrability*. Hence we construct a *one-parameter family* of Bethe-ansatz solutions that interpolate between intersecting and non-intersecting strings.

In this way, we consider two cases: i) Free bundles consisting of N identical strings interacting via identical pair potentials. For this case, we find that the bundle undergoes a unique unbinding transition which is characterized by universal *i.e.*, N-independent critical exponents. We also calculate the density profile of the strings for $N = 2$, 3 and 4 and within a mean-field approximation. For large N, the density profile seems to converge towards the mean-field profile, see fig. 1*a*) below. ii) Bundles interacting with a rigid wall. Extending the corresponding Bethe ansatz for intersecting strings due to Kardar [13], we find a complex phase diagram, see fig. 2 below. Depending on the relative strength of the string-string and the string-wall interactions, the bundle may unbind from the wall via a unique transition or via a sequence of transitions. In all cases, the critical exponents are universal, *i.e.* independent of N.

The free bundle consists of N strings with identical stiffness K. The strings are infinitely extended and run, on average, parallel to the x-direction. Their configurations are parametrized by the displacement fields $l_n(x)$ with $n = 1, \ldots, N$. The effective Hamiltonian for the bundle is given by

$$\mathscr{H}\{l_1(x), \ldots, l_N(x)\} = \int \mathrm{d}x \left\{ \frac{1}{2} K \sum_{n=1}^{N} \left(\frac{\mathrm{d}l_n}{\mathrm{d}x} \right)^2 + \sum_{i>j} V(l_i - l_j) \right\}, \tag{1}$$

where $V(l_i - l_j)$ is the interaction potential for the string pair with $n = i$ and $n = j$. Since this model is 1-dimensional, it can be studied by transfer matrix methods. In the limit of vanishing small-distance cut-off, one obtains a Schrödinger-type equation $\hat{H}\psi(\{l_n\}) = E\psi(\{l_n\})$ with the Hamilton operator

$$\hat{H} = -\frac{T^2}{2K} \sum_{n=1}^{N} \frac{\partial^2}{\partial l_n^2} + \sum_{i>j} V(l_i - l_i).$$

The strings interact with an attractive contact interaction $V_0(l) \equiv -v_0 \delta(l)$ with $v_0 > 0$. In order to model the behaviour of non-intersecting strings the potential

$$V_1(l) \equiv \frac{1}{\varepsilon} v_1 \delta(l + \varepsilon) - \frac{1}{\varepsilon} \frac{v_1}{1 + v_1 K/T^2} \delta(l), \qquad \text{with } v_1 > 0 \tag{2}$$

is added. This additional potential does not suppress all intersections, but favours a specific ordering of the strings. Therefore V_1 leads to partially intersecting strings. The parameter v_1 controls the degree of asymmetry in the probability distribution of the separation variables $l_i - l_j$. In the limit of zero ε, the potential $V_0 + V_1$ leads to a pair of matching conditions for the wave function ψ and its first derivative ψ'. Both functions are discontinuous at $l_i - l_j = 0$; the height of the jump depends on $g_0 \equiv v_0 K/T^2$ and $g_1 \equiv v_1 K/T^2$.

Solving the Schrödinger equation for these matching conditions yields a localized ground state. The N-particle wave function is obtained by the simple product ansatz $\psi_0(l_1, \ldots, l_N) \sim \prod_{i>j} (1 + g_1 \theta(l_i - l_j)) \exp[-p|l_i - l_j|]$ with the transverse momentum

$$p \equiv g_0 \frac{(1 + g_1)^2}{1 + (1 + g_1)^2}. \tag{3}$$

For $g_1 = 0$, one recovers the ground state for N intersecting strings [12]. The preferred ordering imposed to the strings for $g_1 > 0$ is $l_1 < \ldots < l_N$. In the limit of infinite g_1, the wave

function vanishes for all other string permutations resembling, thus, the wave function of non-intersecting strings. The results presented in the following depend on the asymmetry parameter g_1 only through the transverse momentum p. Therefore they remain unchanged in the limit of non-intersecting strings, *i.e.* for infinite g_1, where p is given by $p = g_0$ as follows from (3).

The free energy per unit length, $f(N) = E_0$, is given by

$$f(N) = -\frac{1}{6} N(N^2 - 1) p^2 \frac{T^2}{K} . \tag{4}$$

In the limit of large g_1 this expression agrees with the result in [9]. We now introduce a new set of variables $\{l_1, \ldots, l_N\}$. For a given permutation σ of the strings with $l_{\sigma(1)} < \ldots < l_{\sigma(N)}$, l_n is given by $l_n \equiv l_{\sigma(n)}$. In the limit of large v_1, *i.e.* of non-intersecting strings, the mean position $\langle l_n \rangle$ is equal to the mean position in the original variables, $\langle l_n \rangle$. The wave function ψ_0 is translationary invariant; the mean position $\langle l_n \rangle$ is, therefore, calculated keeping $l_1 = 0$ fixed. The mean extension of the bundle is then given by $l_{\mathrm{bu}} \equiv \langle l_N \rangle - \langle l_1 \rangle \approx \ln(N)/Np$ for large N. The mean separation between neighbouring strings behaves as $\Delta l_n \equiv \langle l_{n+1} \rangle - \langle l_n \rangle = (2p(N-n)n)^{-1}$, with $n = 1, \ldots, N-1$. Both mean separations l_{bu} and Δl_n are characterized by the critical behaviour

$$l_{\mathrm{bu}} \sim \Delta l_n \sim p^{-\psi} , \qquad \text{with } \psi = 1 . \tag{5}$$

The continuum description used here is justified as long as the mean separation between the inner strings is greater than the string thickness $a_\perp$, which implies $N \ll \sqrt{2/a_\perp p}$.

The string density $\varrho_N(l) \equiv \langle \psi_0 | \sum_{n=1}^{N} \delta(l - l_n) | \psi_0 \rangle$ has been calculated for $N = 2, 3, 4$ (where the centre-of-mass coordinate was set equal to zero), see fig. 1*a*). The results can be written in the scaling form $\varrho_N(l) = 2pN^2 \Omega_N(2pNl)$, with the scaling function $\Omega_N(z) = \sum_{j=1}^{N-1} a_{N,j} \cdot \exp[-j|z|]$, where $a_{N,1} = (N-1)/N$, $a_{3,2} = -1/3$, $a_{4,2} = -3/5$, and $a_{4,3} = 3/20$. Note that $\int dl\, \varrho_N(l) = N$ implies $\int dz\, \Omega_N(z) = 1$. The mean-field density defined by $\varrho_{\mathrm{MF}}(l = \langle l_n \rangle -$

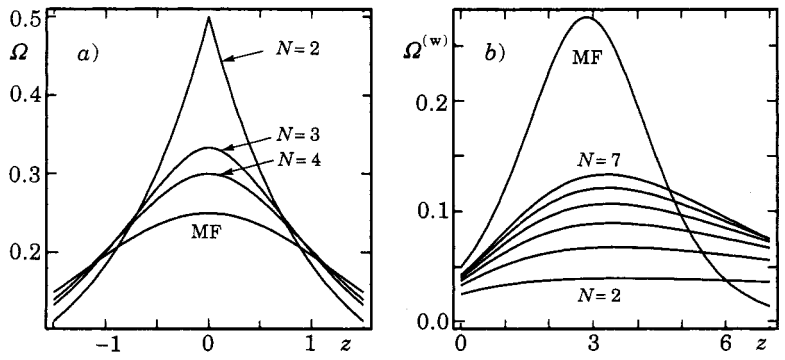

Fig. 1. – *a*) Density profiles for a free bundle of strings: the exact densities $\Omega_N \sim \varrho_N$ as a function of separation $z \sim l$ for $N = 2, 3, 4$ strings together with the mean-field density Ω_{MF}. For increasing N, the exact densities seem to converge towards the mean-field profile. *b*) Density profile for a string bundle interacting with a wall: the exact densities $\Omega_N^{(\mathrm{w})}$ as a function of separation $z \sim l$ for $2 \leqslant N \leqslant 7$ together with the mean-field profile $\Omega_{\mathrm{MF}}^{(\mathrm{w})}$ for $y = -0.95$. The transition from the adhering bundle, regime (B_N), to the free-bundle regime (FB) is located at $y_c = -1$.

$-(1/2)\,l_{\mathrm{bu}}) \simeq /\Delta l_n$ has, in the limit of large N, an analogous scaling form with the scaling function

$$\Omega_{\mathrm{MF}}(z) = (2\cosh(z/2))^{-2}\,. \tag{6}$$

This mean-field density is identical with the density obtained by Helfrich in [8]. For large N, the exact densities $\Omega_N(z)$ seem to converge towards the mean-field profile, see fig. 1*a*).

The unbinding of N strings can be understood heuristically in the framework of an N-state model [10]. If one makes the plausible assumption that locally bound triplets and higher-order multiplets of strings are less likely than locally bound pairs, one finds that the unbinding of the bundle is governed by the unbinding of string pairs. This explains that the unbinding temperature does not depend on N as has been observed in numerical studies for $N = 3$ and $N = 4$ [7, 6]. On the other hand, these numerical studies lead to an N-dependent effective critical exponent ψ. The total interaction potential studied numerically contains only contributions from neighbouring pairs of strings. In contrast, the total interaction potential studied here contains contributions from *all* pairs. The difference corresponds to an effectively repulsive 3-string interaction. Such an interaction represents a marginally irrelevant perturbation and, therefore, leads to large corrections to scaling which should explain the N-dependence found in the numerical work [11].

Next, consider the unbinding of the string bundle from a wall. The N-string system now experiences an additional external potential consisting of an attractive well and a hard wall. This potential causes the wave function of the adjacent string to fall off as $\exp[-ql_1]$ (the possibility that more than one string is in the well is ignored). The ground-state wave function which is analogous to the solution for intersecting strings [13] is of the form

$$\psi_0 \sim \prod_{i>j}(1 + g_1\theta(l_i - l_j))\prod_n \exp[-p_n^{(\mathrm{w})} l_n]\,, \tag{7}$$

with the transverse momenta $p_n^{(\mathrm{w})} \equiv q + 2(n-1)p$, where p is still given by (3). The free energy per unit length of N strings bound to the wall is

$$f^{(\mathrm{w})}(N) = E_0^{(\mathrm{w})} = -\frac{1}{2}N(q + (N-1)p)^2\,\frac{T^2}{K} + f(N)\,, \tag{8}$$

where $f(N)$ is the free energy per unit length of the bundle (4). The mean separation between neighbouring strings is $\Delta l_n = [2(N-n)(q + (N-1+n)p)]^{-1}$.

The state of the system is determined by three parameters: i) the parameter p of the string-string interaction; ii) the transverse momentum q resulting from the string-wall interaction; and iii) the total number N of strings. In fig. 2 the phase diagram is displayed for $N = 5$. In regime (B_N) given by i) $q > -(N-1)p$ for $p > 0$ and ii) $q > -2(N-1)p$ for $p < 0$, all N strings are bound to the wall. For $p > 0$, all N strings unbind simultaneously from the wall at $q = -(N-1)p$. In regime (FB) with $q < -(N-1)p$ and $p > 0$ the strings form a free (or unbound) bundle. For $p < 0$, the strings unbind successively. The n-th string (counted from the wall) peels from the wall at $q = -2(n-1)p$. For $p < 0$ and $0 < q < -2(N-1)p$, we therefore find $N-1$ different regimes (B_n) with $1 \leqslant n \leqslant (N-1)$ strings bound to the wall and the remaining strings completely unbound. For $q < 0$ and $p < 0$, one has the free-string (FS) regime. When the point $p = q = 0$ is approached from regime (B_N), the strings unbind simultaneously from the wall as well as from each other.

Across all phase boundaries, both the free energy $f^{(\mathrm{w})}$ and its first derivative $\partial f^{(\mathrm{w})}/\partial q$ are continuous whereas the second derivative $\partial^2 f^{(\mathrm{w})}/\partial q^2$ exhibits a discontinuity. Hence these transitions are of second order with the critical exponent $\alpha = 0$ for the specific heat. At the phase boundary between (B_N) and (B_{N-1}) the mean separation Δl_n with $n < N$ are

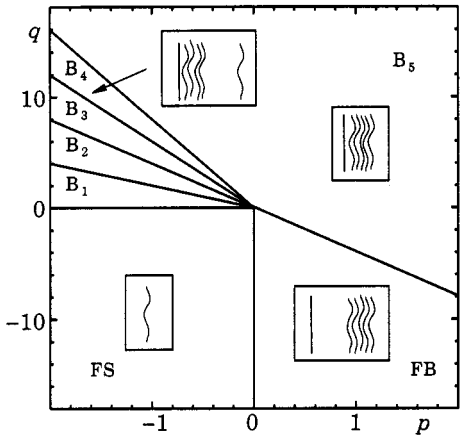

Fig. 2. – Phase diagram for a bundle of $N = 5$ strings interacting with a wall. The parameter p measures the string-string interaction, the parameter q the string-wall interaction. In regime (B_n), n strings are bound to the wall with $N - n$ strings diffusing freely. In the free-bundle regime (FB) the bundle is unbound with respect to the wall. The strings are completely unbound in the free-string regime (FS).

continuous but their first derivatives $\partial \Delta l_n / \partial q$ exhibit a jump. The critical behaviour of the diverging length scales is the same at each transition line, even at the point $p = q = 0$, with the N-independent exponent $\psi = 1$.

In a real system, regime (B_N) and regime (FS) are attained at sufficiently low and sufficiently high temperatures, respectively. Depending on the relative strength of the string-string and the string-wall attraction, the temperature trajectory will mowe from (B_N) to (FS) via the free-bundle regime (FB) or the intermediate states (B_n). In the latter case one has a sequence of N unbinding transitions. In the limit of infinite N, the sequence of critical temperatures $T_c(n)$ attains a finite value $T_c(\infty)$ [6]. It now follows from the explicit expression for the phase boundaries that $1 - (T_c(\infty)/T_c(n))^2 \sim 1/n^\lambda$, with $\lambda = 1$.

For a bundle in regime (B_N) the exact density has the scaling form $\varrho_N^{(w)}(l) = 2pN^2 \Omega_N^{(w)}(2pNl, q/p(N-1))$. The scaling functions $\Omega_N^{(w)}(z, y)$ are displayed in fig. 1*b*) for $2 \leqslant N \leqslant 7$ and $y = -0.95$. Note that the phase boundary between regime (B_N) and regime (FB) is located at $y_c = -1$. The mean-field density defined by $\varrho_{MF}^{(w)}(l = \langle l_n \rangle) \simeq 1/\Delta l_n$ has the scaling form $\varrho_{MF}^{(w)}(l) = 2pN^2 \Omega_{MF}^{(w)}(2pNl, q/Np)$, with

$$\Omega_{MF}^{(w)}(z, y) = (1 + y/2)^2 [\cosh((1 + y/2)(z - z_{max}))]^{-2}, \tag{9}$$

and $z_{max} \equiv -(2 + y)^{-1} \ln(1 + y)$. The unbinding transition towards the (FB) regime is not correctly described by the mean-field density. As y approaches $y_c = -1$ from above, the mean separation of the bundle from the wall scales as $\sim \ln(1/|y - y_c|)$ within the mean-field theory, whereas the correct critical behaviour is given by $\sim |y - y_c|^{-\psi}$, with $\psi = 1$. At the transition from regime (B_N) to regime (B_{N-1}), *i.e.*, at $p = -q/2(N-1)$ the mean-field density exhibits a power law tail which is given by

$$\varrho_{MF}^{(w)}(l) \approx N l_{sc}/(l_{sc} + l)^2 \quad \text{for large } N, \tag{10}$$

where $l_{sc} \equiv 1/q$. As before the continuum limit is only justified as long as the string separation Δl_n is larger than the string thickness $a_\perp$. This leads to the crossover scale l_* defined by $\varrho_{MF}^{(w)}(l = l_*) = 1/a_\perp$. The strings with $l < l_*$ are densely packed; the strings with

$l > l_*$ are swollen and should exhibit the power law tail as in (10). The swollen region contains $n_{sw} \sim (N/qa_\perp)^{1/2}$ strings.

In summary, we have obtained analytic results on the unbinding transition and density profiles of bundles of $(1+1)$-dimensional strings both for a free bundle and for a bundle interacting with a rigid wall. The critical behaviour we have found here should apply to all pair potentials which decay faster than l^{-2} for large l and hence belong to the strong-fluctuation regime. In general, many-string forces are present as well; for example, the force between two strings may be screened by a third string in between. Renormalization group arguments [11] as well as the scaling picture of [10] show that such screening forces do not alter the asymptotic scaling. With large *attractive* many-string forces, however, the transition is governed by a different fixed point, and the Bethe ansatz breaks down [11]. Further, the critical behaviour of fluid membrane bunches should be analogous to the behaviour of string bundles. Therefore, the asymptotic critical behaviour for N identical membranes should be governed by universal critical exponents. Another interesting problem are strings interacting via short-ranged pair potentials in $d = 1 + d_\perp$ dimensions. For $d_\perp < 4$, two strings exhibit continuous unbinding transitions [14], and one would expect that the associated critical behaviour also applies to string bundles with $N > 2$. An explicit calculation for this system would be quite valuable.

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We thank F. JÜLICHER and R. NETZ for stimulating interactions.

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