

# From bunches of membranes to bundles of strings

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**Abstract.** Bunches of membranes and bundles of strings exhibit unbinding transitions from a bound state at low temperatures to an unbound state at high temperatures.  $N$  freely suspended manifolds unbind continuously at the unique unbinding temperature  $T_u^f$  which is independent of  $N$ . The amplitudes of the critical singularities have a strong  $N$ -dependence, however, which implies that the critical region for the continuous transition becomes very small and the transition becomes very abrupt in the limit of large  $N$ . If  $N$  membranes or strings are bound to a rigid surface, they undergo a sequence of either two or of  $N$  successive transitions. In general, the rigid surface affects the contact probabilities of the fluctuating manifolds. For effectively repulsive interactions, the contact exponent  $\zeta_2$  which governs the probability for local pair contacts satisfies the scaling relation  $\zeta_2 = d_{||} + \zeta$  where  $d_{||}$  and  $\zeta$  denote the dimensionality and the roughness exponent of these manifolds.

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## 1. Introduction and overview

Lipid bilayers in solution often form bunches in which several membranes are, on average, parallel to each other. Two different geometries will be considered: (i) Freely suspended bunches as shown in Fig. 1(a), and (ii) Bunches which adhere to a rigid surface or wall, see Fig. 1b. The latter geometry is obtained, e.g., by spreading a concentrated lipid solution on a glass slide.

The structure of such a bunch can be characterized by its density profile which depends on the mean separations of the membranes within the bunch. If the shape fluctuations of these membranes are strong, they drive the membranes apart and lead to loosely bound or highly swollen states. In fact, membranes which are bound together by attractive van der Waals forces become completely unbound at sufficiently high temperatures and, thus, undergo unbinding transitions at a characteristic unbinding temperature [1].

In the following, I will briefly review recent theoretical work on these unbinding transitions. The main emphasis is on the dependence of the critical behavior on the number of membranes within the bunch. For a bunch of  $N$  freely suspended membranes, the transition from the bound to the unbound state proceeds via a *unique* unbinding transition, see Sect. 3 below. In fact, the unbinding temperature  $T_u^f$  of the freely suspended bunch turns out to be independent of  $N$ . [2, 3]

In the case of  $N$  membranes attracted towards a rigid surface or wall, the transition depends on the relative strength of the surface-membrane and the membrane-membrane attraction as explained in Sect. 4. There are essentially two possibilities: [3, 4] (i) If the attraction towards the substrate is sufficiently weak, one has a sequence of two transitions: the whole  $N$ -bunch unbinds from the substrate at the unbinding temperature  $T = T_u^b < T_u^f$  and then undergoes the transition towards completely unbound membranes at  $T = T_u^f$ ; and (ii) If the attraction towards the rigid surface is relatively strong, one has a sequence of  $N$  unbinding transitions and thus a sequence of unbinding temperatures  $T_u^n$  with  $T_u^f < T_u^n \leq T_u^l$ . At each of these transitions, a single membrane peels off from the bunch.

From a conceptual point of view, it is very useful to consider the case of membranes which do not intersect and thus repel each other by repulsive hard wall interactions but are kept together by an external pressure. This is described in Sect. 3.1 and Sect. 4.1 for freely suspended membranes and for membranes at a substrate, respectively. A precise definition of the effective interactions which arise from the renormalization by the shape fluctuations is given in the Appendices A and B at the end of this paper.

Unless stated otherwise, the membranes considered here are tacitly assumed to be in a *fluid* state and thus to have vanishing shear modulus. The scaling properties of these membranes are very similar to those of 1-dimensional strings. In the context of condensed matter, the term string refers to a fluctuating line governed by a finite line tension; one example is provided by stretched (or directed) polymers. At the end, I will briefly mention that the analogy between membranes and strings breaks down in certain cases.

Dedicated to Herbert Wagner on the occasion of his 60th birthday

Even though this paper is basically a review, it contains a few original pieces: the  $N$ -dependence of the effective interaction between the outmost manifolds of the bunch or bundle is carefully discussed in Sect. 3.1; a new scaling relation for the contact exponent is derived, see (3.7); it is argued that the unbinding transition of freely suspended bunches becomes very abrupt and thus looks discontinuous for large  $N$  even though it is still continuous, see (3.13); and the contact probabilities close to a rigid surface are shown to exhibit a different scaling behavior, see (4.1).

## 2. Generic interactions of membranes

The separation of the membranes is governed by their effective interaction. This interaction is determined by the interplay of direct interactions arising from the forces between the molecules and fluctuation-induced interactions arising from the shape fluctuations of the membranes.

*Molecular forces and direct interactions.* The direct interaction  $V(l)$  between two rigid membranes at separation  $l$  can be measured by the surface force apparatus consisting of two mica surfaces onto which the membranes are immobilized. [5] The simplest example is provided by lipid bilayers which (i) are electrically neutral, and (ii) interact across a water layer which contains no macromolecules or colloids. In this case, the interaction potential  $V(l)$  is composed of a repulsive hydration and an attractive van der Waals interaction and has the schematic form as shown in Fig. 2a.

Lipid bilayers may become charged by adsorption of ions from the solution or by dissociation of their head groups. They then exhibit electric double layers which usually lead to *repulsive* interactions between the surfaces as predicted by the classical Poisson-Boltzmann theory. The combination of van der Waals and electrostatic interactions often leads to a potential barrier as shown in Fig. 2b.

If the two rigid membranes are subject to an external pressure  $P$ , one has  $-\partial V/\partial l = P$ , i.e., the disjoining pressure,  $-\partial V/\partial l$ , arising from the direct interaction is balanced against the external pressure  $P$

*Fluctuation-induced interactions.* Membranes immersed in a liquid solution will undergo thermally-excited shape fluctuations. On scales which are large compared to the size of the molecules, the typical shape fluctuations should be bending modes or undulations in which the surface area of the membrane remains unchanged. Bending undulations of wavelength  $L_{\parallel}$  are characterized by the roughness  $L_{\perp} \sim (T/\kappa)^{1/2} L_{\parallel}$  where  $T$  and  $\kappa$  denote the temperature (in energy units) and the bending rigidity of the membrane, respectively. [6]

In the presence of another surface at separation  $l$ , the undulating membrane has less configurational entropy. Indeed, all fluctuations of the free membrane which exceed a certain wavelength  $\xi_{\parallel}$  are strongly suppressed by the second surface. On the other hand, those fluctuations with a wavelength below  $\xi_{\parallel}$  remain essentially unaffected by the confinement. If both surfaces are flexible with bending rigidities  $\kappa$  and  $\kappa'$ ,

the fluctuations of their separation  $l$  is governed by the effective bending rigidity  $\kappa_{\text{eff}} = \kappa\kappa'/(\kappa + \kappa')$ . One is thus led to consider an ideal gas of uncorrelated membrane segments which have a longitudinal size  $\xi_{\parallel}$  and a transverse extension  $\xi_{\perp} \sim (T/\kappa_{\text{eff}})^{1/2} \xi_{\parallel}$ . The latter length scale  $\xi_{\perp}$  is usually of the order of the mean membrane separation  $l \equiv \langle l \rangle$

One may now allude to the equipartition theorem and postulate that each such hump has a free energy  $\Delta\mathcal{F} \sim T$ . This implies that the hump free energy per unit (projected) area behaves as [7]

$$V_{fl} \equiv \Delta\mathcal{F}/\xi_{\parallel}^2 \approx c_{fl} T^2/2\kappa_{\text{eff}} \ell^2 \quad \text{for large } \ell. \quad (2.1)$$

This free energy per unit area represents a fluctuation-induced interaction between the membranes which leads to the disjoining pressure  $P = -\partial V_{fl}/\partial \ell \sim 1/\ell^3$ . For two identical membranes with  $\kappa' = \kappa$ , one has  $2\kappa_{\text{eff}} = \kappa$ .

*From membranes to strings.* The scaling arguments just described for membranes may also be applied to 1-dimensional strings with line tension  $\sigma$ . As a result, one arrives at the fluctuation-induced interaction  $V_{fl}(\ell) \sim T^2/\sigma_{\text{eff}} \ell^2$  as was first derived in the context of commensurate-incommensurate transitions [8] and is implicit in some earlier work about steps on crystal surfaces [9].

Comparison with the relation (2.1) shows that the fluctuation-induced interaction  $V_{fl}$  for strings has the same functional dependence on the mean separation  $\ell$  as for fluid membranes. Thus, one expects that the shape fluctuations of strings and of fluid membranes have a rather similar effect on the direct interaction of these objects. This is also predicted by functional renormalization which leads to the same fixed point potentials for the unbinding transitions of strings and membranes. [10] It will become clear below that this analogy should indeed hold as long as the direct interaction  $V(l)$  exhibits a single minimum as in Fig. 2a.

## 3. $N$ freely suspended membranes or strings

Now, consider a bunch of  $N$  identical membranes which are freely suspended in solution as in Fig. 1a. All membranes are taken to have the same bending rigidity and each adjacent pair of membranes interacts with the direct interaction  $V(l)$ . Two cases will be discussed: (i) Repulsive hard-wall interactions corresponding to non-intersecting membranes which are kept together by an external pressure; and (ii) Interactions with an attractive potential well as in Fig. 2a. In the latter case, the bunch undergoes an unbinding transition at a characteristic unbinding temperature.

### 3.1. Repulsive Hard-wall interaction

First, consider the simple case of membranes which repel each other by repulsive hard wall potentials as given by

$$V_{hw}(l) \equiv \infty \quad \text{for } l < 0 \\ \equiv 0 \quad \text{for } l > 0 \quad (3.1)$$

As explained in Appendix B, these membranes may be kept together by an external pressure  $P$  or by imposed boundaries. These different constraints define different ensembles

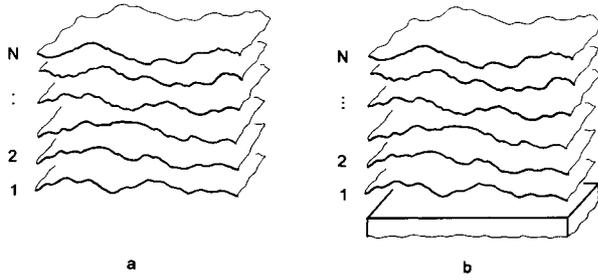


Fig. 1a,b. Two types of membrane bunches: **a** Freely suspended bunch, and **b** Bunch adjacent to a rigid surface

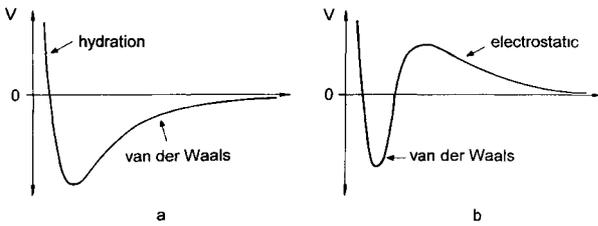


Fig. 2a,b. Direct interaction  $V$  arising from molecular forces as a function of the membrane separation  $l$ : **a** Hydration and van der Waals interaction between electrically neutral membranes; and **b** Hydration, van der Waals and electrostatic interaction between electrically charged membranes leading to a potential barrier

in which one may study the renormalization of  $V_{hw}(l)$  by the shape fluctuations.

*Computer simulations.* If the bunch is kept together by the pressure  $P$ , the fluctuations drive the membranes apart in the limit of small  $P$ . This unbinding process has been studied by Monte Carlo simulations for  $N = 2, 3$  and  $4$ .

Here and below, the mean separation of the two outer membranes will be denoted by  $\ell_{1N}$ . The mean separation  $\ell_n$  of two adjacent membranes labelled by  $(n)$  and  $(n+1)$  within the bunch is expected to be of the order of  $\bar{\ell} \equiv \ell_{1N}/(N-1)$ . In fact, in the limit of small  $P$ , all separations  $\ell_n$  are found to become equal to  $\bar{\ell}$ , and  $\bar{\ell}$  is found to be independent of  $N$  within the numerical accuracy of the simulations.

The behavior of  $\bar{\ell}$  as a function of  $P$  determines the fluctuation-induced interaction  $V_{fl}(\bar{\ell})$  via  $P = -\partial V_{fl}(\bar{\ell})/\partial \bar{\ell}$ , see (B.2) in Appendix B. In all cases studied numerically, one finds that

$$V_{fl}(\bar{\ell}) \approx c_{fl}^{(N)} T^2/\kappa \bar{\ell}^2 \quad \text{for large } \bar{\ell}. \quad (3.2)$$

For  $N = 2$ , the data of [11] imply  $c_{fl}^{(2)} = 0.115 \pm 0.005 \simeq 3\pi^2/256$ . For  $N = 3$  and  $N = 4$ , the Monte Carlo data lead to the estimate  $c_{fl}^{(3)} = 0.113 \pm 0.005$  and  $c_{fl}^{(4)} = 0.111 \pm 0.006$  respectively. [12, 13] Thus, within the numerical accuracy of these estimates, these three values are identical.

Another geometry which has been studied by MC simulations are  $N$  identical membranes with bending rigidity  $\kappa$  confined between two rigid walls. [14, 15] The separation of the two rigid walls was taken to be  $(N+1)\bar{\ell}$  so that the mean separations of all nearest neighbor surfaces are of the order of  $\bar{\ell}$ . As explained in Appendix B, the free energy

density within this fixed slab ensemble leads to another definition of the fluctuation-induced interaction  $V_{fl}$ . The dimensionless coefficient  $c_{fl}^{(N)}$  found from these simulations was  $c_{fl}^{(1)} \simeq 0.080$ ,  $c_{fl}^{(3)} \simeq 0.093$ , and  $c_{fl}^{(5)} \simeq 0.097$  for  $N = 1, 3$  and  $5$  membranes. [14] Extrapolation of these data to large  $N$  gave the estimate  $c_{fl}^{(\infty)} \simeq 0.106$  which is somewhat smaller than but comparable with the value  $c_{fl} \simeq 0.115 \simeq 3\pi^2/256$  as obtained in the pressure ensemble.

In [7], Helfrich had originally obtained three different estimates for the coefficient  $c_{fl}$ . One of these estimates was  $c_{fl} = 3\pi^2/128$  which is exactly twice the value as obtained from the pressure ensemble. The value  $c_{fl} = 3\pi^2/128$  was also deduced experimentally from X-ray scattering data for lamellar phases in oil-water-surfactant mixtures. [16] At present, there is no explanation for this discrepancy.

*Bundles of strings.* A bundle of  $N$  strings in two dimensions is equivalent to  $N$  quantum-mechanical particles. If the strings experience hard wall potentials corresponding to the non-intersecting constraint, these particles are *free fermions*. [17] One convenient way to confine the thickness of the string bundle is to put them onto a strip of width  $L_{\perp}$  and to use periodic or fixed boundary conditions in the perpendicular direction.

The free energy of such a string system is easily calculated since it is equal to the quantum-mechanical energy of the corresponding free fermion ground state. This free energy can again be used to derive an explicit expression for the fluctuation-induced interaction  $V_{fb}$ , see Appendix B. For both types of boundary conditions, one obtains  $V_{fb}(\bar{\ell}) = c_{fb}^{(N)} T^2/\sigma \bar{\ell}^2$  but with somewhat different coefficients  $c_{fb}^{(N)}$ . For periodic boundary conditions, one has

$$\begin{aligned} c_{fb}^{(N)} &= (\pi^2/6)(1 - 1/N^2) & \text{for odd } N \\ &= (\pi^2/6)(1 + 2/N^2) & \text{for even } N \end{aligned} \quad (3.3)$$

For fixed boundary conditions, on the other hand, one finds [14]

$$c_{fb}^{(N)} = (\pi^2/6)(2N+1)/(2N+2). \quad (3.4)$$

Thus, for large  $N$ , both boundary conditions lead to the limiting value  $c_{fb}^{(\infty)} = \pi^2/6 \simeq 1.64$ .

For  $N = 2$ , one can also calculate the free energy when the strings are confined by the external pressure  $P$  which leads to the coefficient  $c_{fb}^{(2)} \simeq 1.89$ . The corresponding values for  $N > 2$  have not been calculated in the pressure ensemble but it seems plausible that the limiting value for large  $N$  is again given by  $c_{fb}^{(\infty)} = \pi^2/6$  as for periodic and fixed strips. Thus, for strings in the pressure ensemble, the coefficients  $c_{fb}^{(N)}$  should vary from  $c_{fb}^{(2)} \simeq 1.89$  down to  $c_{fb}^{(\infty)} \simeq 1.64$ , i.e., by about 15% which should be clearly visible in numerical studies.

*Contact probabilities.* In the case of hard wall interactions, the heuristic scaling picture described in Sect. 2 above leads to the correct  $\ell$ -dependence of the renormalized interaction.

One must be careful, however, not to draw wrong conclusions from this picture. For example, it is tempting to use this picture in order to estimate the contact probability of two adjacent manifolds. Since the scaling picture suggests one collision per  $\xi_{\parallel}$  hump, one would conclude that the probability  $\mathcal{P}_2$  for local pair contacts scales as  $1/\ell^2$  both for membranes and for strings.

This is, however, wrong. For two interacting strings, the probability distribution for the separation  $l$  is equal to the squared wave function of the ground state which vanishes as  $l^2$  for small  $l$ . This implies that the contact probability  $\mathcal{P}_2$  vanishes as  $\mathcal{P}_2 \sim 1/\ell^{2\zeta_2}$  for large  $\ell$  with the contact exponent  $\zeta_2 = 3/2$ . [18, 19] Thus, 2-string collisions or locally bound pairs of strings are less frequent than suggested by the scaling picture.

For bundles of non-intersecting strings corresponding to free fermions, one may calculate the contact probability  $\mathcal{P}_n$  for  $n$ -string collisions with  $n > 2$ . Using the confinement by a periodic strip of width  $L_{\perp}$ , one finds

$$\mathcal{P}_n \sim 1/L_{\perp}^{2\zeta_n} \sim 1/\ell^{2\zeta_n} \quad \text{with} \quad \zeta_n = (n^2 - 1)/2 \quad (3.5)$$

from the ground state wave function of the free fermion system. [20, 21]

Using the analogy between strings and membranes, one predicts that the contact probability  $\mathcal{P}_n$  for  $n$ -membrane collisions behaves as  $\mathcal{P}_n \sim 1/\ell^{\zeta_n}$  with  $\zeta_n = n^2 - 1$ . The value  $\zeta_2 = 3$  has been recently confirmed by Monte Carlo simulations. [22]

The contact exponents  $\zeta_n$  as given by (3.5) can also be derived from a careful interpretation of the necklace model for  $n$  identical strings. [23, 21] In this model, one considers all configurations of  $n$  strings which consist of alternating segments, say A and B: In the A segments, all  $n$  strings are strongly bound together; in the B segments, all  $n$  strings are unbound from one another. [24, 25] This is equivalent to a bundle of strings which experience repulsive hard wall potentials between nearest neighbors and an attractive short-ranged  $n$ -body potential. The critical behavior within this model can be understood by mapping it onto the intermediate fluctuation regime for two strings. [23] If the short-ranged  $n$ -body potential is not able to bind the bundle together, their contact probabilities are again given by (3.5). [21]

The necklace model for three strings can also be solved if the strings have different line tensions  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  [26, 27] This model can be mapped onto a random walk in a wedge geometry where the wedge angle  $\theta$  satisfies  $0 \leq \theta \leq \pi/2$  and  $\tan(\theta) = \sqrt{(\sigma_2/\sigma_1) + (\sigma_2/\sigma_3) + (\sigma_2^2/\sigma_1\sigma_3)}$ . If the 3-body potential is not able to bind the strings together, the contact exponent  $\zeta_3$  for three-string collisions is given by [21]

$$\begin{aligned} \zeta_3(\theta) &= 1 + \pi/\theta \\ &= 1 + \pi/\arctan \left[ \sqrt{(\sigma_2/\sigma_1) + (\sigma_2/\sigma_3) + (\sigma_2^2/\sigma_1\sigma_3)} \right] \end{aligned} \quad (3.6)$$

which depends continuously on the ratios of the line tensions.

*Effective interaction of the two outer manifolds.* Let us now focus on the two outer manifolds of the bunch or bundle and

let us consider the effective interaction between these two outer manifolds arising from the confinement of the  $N - 2$  interior ones. As before, the mean separation of the two outer membranes is denoted by  $\ell_{1N}$

It is intuitively clear that the effective interaction arising from the intermediate manifolds must be repulsive. Within the heuristic scaling picture, it seems plausible to assume that the effective interaction  $V_{1N}$  between the two outer manifolds decays as  $1/\ell_{1N}^2$ . I now want to point out, however, that *the strength of this effective interaction cannot be defined in a unique way.*

First, let us identify the effective interaction  $V_{1N}$  with the renormalized interaction  $F(\ell_{1N})$  as defined in Appendix A, see (A.4). This implies the disjoining pressure  $P = -\partial V_{1N}/\partial \ell_{1N} = -\partial V_{f_l}(\bar{\ell})/\partial \bar{\ell}$  with  $\ell_{1N} = (N - 1)\bar{\ell}$  and  $V_{1N} = (N - 1)V_{f_l}(\bar{\ell})$ . Using  $V_{f_l}(\bar{\ell}) \approx c_{f_l}^{(2)}T^2/\sigma\bar{\ell}^2$ , one has  $V_{1N} \approx c_{1N}T^2/\sigma\ell_{1N}^2$  with  $c_{1N} = (N - 1)^3c_{f_l}^{(2)}$  and thus  $c_{1N} \sim N^3$  for large  $N$ . The latter  $N$ -dependence also follows from the free energy of  $N$  strings confined to a strip of width  $L_{\perp}$

On the other hand, one may also focus on the contact probability  $\mathcal{P}_N$  for  $N$ -string collisions. This probability will now be identified with the probability  $\mathcal{P}_2$  for the two outer strings which interact via the effective interaction  $V_{1N} \approx W/\ell_{1N}^2$  with  $W > 0$ . For sufficiently large  $W$ , such an interaction belongs to subregime (C) of the intermediate fluctuation regime for two manifolds. [10] In the latter case, one obtains the contact probability  $\mathcal{P}_2 \sim 1/\ell_{1N}^{2\zeta_2(w)}$  with  $\zeta_2(w) = 1 + (w + 1/4)^{1/2}$  and  $w \equiv \sigma W/T^2$ . If this is set equal to the probability  $\mathcal{P}_N$  for  $N$ -string collisions of the whole bunch, one has  $\zeta_2(w) = \zeta_N = (N^2 - 1)/2$  which is an implicit equation for the potential strength  $W = T^2w/\sigma$ . It then follows that  $V_{1N} \approx c_{1N}T^2/\sigma\ell_{1N}^2$  with  $c_{1N} = [(N^2 - 3)^2 - 1]/4$  which is the same expression as for the necklace model of  $N$  strings [23].

Note that the coefficient  $c_{1N}$  as obtained from the contact probability  $\mathcal{P}_N$  behaves as  $c_{1N} \sim N^4$  for large  $N$  in contrast to the behavior  $c_{1N} \sim N^3$  as obtained from the disjoining pressure. Therefore, the *global* behavior of the two outer manifolds as described by their mean separation  $\ell_{1N}$  and their *local* behavior as described by their contact probability  $\mathcal{P}_N$  cannot be derived from the same effective interaction.

*Digression: Scaling relation for the contact exponent.* In general, the fluctuations of a  $d_{\parallel}$ -dimensional manifold in  $d = d_{\parallel} + 1$  dimensions are characterized by the roughness exponent  $\zeta$ . The fluctuation-induced interaction  $V_{f_l}(l)$  then decays as  $\sim 1/l^{\tau}$  with  $\tau = d_{\parallel}/\zeta$  for thermally-excited fluctuations as considered here, and one has  $\ell \sim 1/P^{\psi}$  for small  $P$  with  $\psi = 1/(1 + \tau) = \zeta/(d_{\parallel} + \zeta)$ . In the context of the renormalization group, this follows from the fact that the term  $Pl$  represents a relevant perturbation which transforms according to  $P \rightarrow P' = b^{\lambda_P}P$  with scaling index  $\lambda_P = 3$  under renormalization. On the other hand, the contact exponent  $\zeta_2$  which governs the probability for pair contacts is related to short-ranged perturbations such as  $G\delta(l)$  which transforms as  $G \rightarrow G' \approx b^{\lambda_{SR}}G$  with scaling index  $\lambda_{SR}$  under renormalization. More precisely, the contact exponent  $\zeta_2$  is given by the scaling relation  $\zeta_2 = d_{\parallel} - \lambda_{SR}$ . [18, 21]

At the "hard-wall" or purely repulsive fixed point, short-ranged perturbations are irrelevant with  $\lambda_{SR} < 0$

[28, 29] In fact, functional renormalization leads to an irrelevant short-ranged eigenperturbation with scaling index  $\lambda_2 = -\zeta$  which is related to a translation of  $l$  and reflects the choice of origin for  $l$ . [28] If there is no other short-ranged eigenperturbation with scaling index  $\lambda > \lambda_2$ , one has  $\lambda_{SR} = \lambda_2 = -\zeta$  and the contact exponent  $\zeta_2$  satisfies the scaling relation

$$\zeta_2 = d_{||} + \zeta \quad (3.7)$$

for hard-wall interactions in  $d = d_{||} + 1$  dimensions. For membranes in three dimensions governed by lateral tension, one has  $\zeta = 0$  and (3.7) predicts  $\zeta_2 = 2$  as observed in recent Monte Carlo simulations. [30]

### 3.2. Unbinding at a unique transition temperature

Real membranes such as lipid bilayers usually attract each other by van der Waals forces. If the bilayers are electrically neutral, their direct interaction has the form as shown in Fig. 2a. At sufficiently low temperatures, the membranes form a bound state. As the temperature is increased, the shape fluctuations of the membranes grow and lead to unbinding transitions.

Unbinding transitions were first predicted theoretically from functional renormalization group calculations. [1] They were experimentally observed by Helfrich and Mutz for membranes composed of the sugarlipid DGDG. [2] In these experiments, membrane bunches were observed which contained between  $N = 2$  and  $N = 20$  membranes.

Now, one would like to know if the critical phenomena at these unbinding transitions depend on the number of membranes contained in the bunch. It turns out that the transition temperature is independent of the number of membranes but that the critical behavior shows a strong  $N$ -dependence.

*Numerical studies.* The direct interaction  $V(l)$  is now taken to be an attractive square-well potential defined by  $V(l) = \infty$  for  $l < 0$  and by

$$\begin{aligned} V(l) &= -|U| & \text{for } 0 < l < l_v \\ &= 0 & \text{for } l_v < l \end{aligned} \quad (3.8)$$

which has potential range  $l_v$  and potential depth  $|U|$ . For such a potential, completely analogous behavior has again been found numerically for membranes and strings. In this case,  $N = 2$  and 3 membranes have been studied by Monte Carlo simulations and  $N = 2, 3$  and 4 strings by numerical iterations of the transfer matrix.

As a result, one finds that all manifolds of the bunch or bundle unbind simultaneously at a unique transition. The corresponding unbinding temperature,  $T_u^f$ , turns out to be independent of  $N$  (if one uses the same discretization). Over the accessible range of scales, the mean separations  $\ell_n \equiv \langle l_n \rangle$  of the adjacent manifolds labelled by  $(n)$  and  $(n+1)$  with  $n = 1, \dots, N-1$  are well fitted by

$$\ell_n - l_v/2 \approx A/|T - T_u^f|^\psi \quad (3.9)$$

where both the amplitude  $A$  and the effective exponent  $\psi$  decrease with increasing  $N$

The numerical values obtained for the effective exponent  $\psi$  are  $\psi = 0.99 \pm 0.05$  and  $\psi = 0.91 \pm 0.04$  for  $N = 2$  and  $N = 3$  membranes, respectively. For strings, the numerical transfer matrix iterations lead to the effective exponents  $\psi = 1.005 \pm 0.005$  for  $N = 2$  (in agreement with the exact value  $\psi = 1$  for the asymptotic critical behavior) and  $\psi \simeq 0.94$  for  $N = 3$ . For  $N = 4$ , the inner separation  $\ell_2$  is governed by  $\psi = 0.87 \pm 0.02$  and the two outer separations  $\ell_1$  and  $\ell_3$  by  $\psi = 0.93 \pm 0.02$ . Thus, the numerical studies lead to effective exponents which depend both on  $N$  and on  $n$ .

*N-state model for unbinding transition.* The fact that the unbinding temperature  $T_u^f$  does not depend on  $N$  agrees with the experimental observations [2] and can be understood within the following scaling picture [31]. Consider the case  $N = 3$ . Locally, three membranes (or strings) which interact via short-ranged potentials can attain three different types of configurations: (i) All three membranes are 'locally unbound' if their separation exceeds the range of the interaction potential; (ii) two of the three membranes form a bound pair whereas the third one is 'locally unbound'; and (iii) all three membranes form a bound triplet.

The probabilities for these three different local configurations will be denoted by  $\mathcal{P}_{ub}$ ,  $\mathcal{P}_2$ , and  $\mathcal{P}_3$ , respectively, and the corresponding excess free energies per unit area (or unit length) by  $\Delta F_{ub}$ ,  $\Delta F_2$ , and  $\Delta F_3$ , respectively. The excess free energy per unit area of the bunch is then estimated as [31]

$$\Delta F \simeq \Delta F_{ub} \mathcal{P}_{ub} + \Delta F_2 \mathcal{P}_2 + \Delta F_3 \mathcal{P}_3 \quad (3.10)$$

If the unbinding transition is continuous, the contact probabilities  $\mathcal{P}_2$  and  $\mathcal{P}_3$  must vanish in a continuous way whereas  $\mathcal{P}_{ub} \approx 1$  as the transition is approached.

It is convenient to use the extension  $\ell_{13}$  of the bunch or bundle as the basic length scale. For bending undulations, the excess free energy per unit area  $\Delta F_{ub}$  of 'locally unbound' segments then behaves as  $\Delta F_{ub} \sim 1/\ell_{13}^2$ . The leading term of the excess free energy  $\Delta F_2$ , on the other hand, arises from the locally bound pair which is determined by the microscopic interaction parameters. Likewise, the excess free energy  $\Delta F_3$  is also determined by these parameters. For strings interacting via the square-well potentials as defined by (3.8), one has the estimates  $\Delta F_2 \simeq cT^2/\sigma l_v^2 - |U|$  and  $\Delta F_3 \simeq 2\Delta F_2$ . It then follows from  $\Delta F_2 = 0$  that the unbinding temperature  $T_u^f \sim [l_v^2 |U| \sigma]^{1/2}$

Since it is intuitively clear that  $\mathcal{P}_3 \ll \mathcal{P}_2$  (for pair potentials as considered here), one has to balance the first two terms of the expression (3.10). This implies that a continuous transition can only occur if  $\mathcal{P}_2$  decays more slowly than  $1/\ell_{13}^2$ . In addition, it also follows from this balance that *such a continuous unbinding transition occurs when  $\Delta F_2$  vanishes*. Therefore, the unbinding temperature for  $N = 3$  is the same as the one for  $N = 2$

For general  $N$ , one may define  $N$  different sets of configurations. The first two sets again contain (i) locally unbound states, and (ii) states with one locally bound pair and thus one local contact; these states occur with probabilities  $\mathcal{P}_{ub}$  and  $\mathcal{P}_2$ , respectively. The  $n$ -th set with  $n > 2$  is taken to

contain all configurations with  $(n - 1)$  local contacts. The excess free energy of this  $N$ -state model is then given by  $N$  terms corresponding to these  $N$  states. The first two terms have the same form as in (3.10) with  $\Delta F_{ub} \sim 1/\ell_{1N}^2$ . It then follows that the bunch undergoes a continuous transition provided that the contact probability  $\mathcal{P}_2$  decays more slowly than  $1/\ell_{1N}^2$ . If such a transition occurs, the corresponding unbinding temperature is again determined by  $\Delta F_2 = 0$  as for  $N = 2$  and  $N = 3$ .

The scaling behavior of the contact probability  $\mathcal{P}_2$  and thus the critical behavior of  $\ell_{1N}$  for small  $|\Delta F_2|$  cannot be determined within this  $N$ -state model. Several methods have been recently used in order to determine this behavior analytically: (i) Mean-field theories [32, 33]; (ii) Bethe Ansatz solutions for strings with pair interactions [34, 4]; and (iii) Field-theoretic renormalization for strings with many-body interactions [20]. The results of the last two methods will be discussed in the following.

*Strings with pair interactions: Bethe Ansatz solution.* As mentioned,  $N$  strings in two dimensions are equivalent to  $N$  quantum-mechanical particles: intersecting and non-intersecting strings correspond to Bosons and Fermions, respectively. If the strings interact via short-ranged pair potentials, the corresponding Schrödinger equation can be solved analytically for arbitrary  $N$  using the Bethe Ansatz: If one modifies the short-ranged potential, one can even construct a one-parameter family of Bethe Ansatz solutions which interpolates between intersecting and non-intersecting strings [4].

The modified potential is characterized by infinitesimal potential range and by the effective potential depth  $gT^2/\sigma$  which corresponds to the excess free energy  $\Delta F_2$  per unit length of local pair contacts. As mentioned,  $\Delta F_2 \simeq cT^2/\sigma l_v^2 - |U|$  for the square well potential defined by (3.8). It then follows that the parameter  $g \simeq (c - l_v^2 |U| \sigma / T^2) / l_v \sim -|T - T_u^f|$ .

For this model, the mean separation of the two adjacent strings labelled by  $(n)$  and  $(n + 1)$  is given by [4]

$$\ell_n = 1/2n(N - n)|g| \sim 1/n(N - n)|T - T_u^f|^\psi \quad (3.11)$$

with  $\psi = 1$ .

Thus, all length scales diverge with the universal unbinding exponent  $\psi = 1$  but exhibit an amplitude which depends strongly on  $N$ : The outer strings with  $n \sim N$  and the inner strings with  $n \sim N/2$  are characterized by  $\ell_n \sim 1/N$  and by  $\ell_n \sim 1/N^2$ , respectively. Thus, the inner strings are more densely packed than the outer ones (as observed in the numerical studies for  $N = 4$ ). The extension of the whole bundle is found to behave as [4]

$$\ell_{1N} \approx \ln(N)/Ng \sim \ln(N)/N|T - T_u^f| \quad (3.12)$$

for large  $N$  which is dominated by the mean separations of the outer string pairs.

The above expressions for  $\ell_n$  and  $\ell_{1N}$  have been obtained in the limit of infinitesimal potential range and, thus, apply to length scales which are large compared to the microscopic potential range  $l_v$  (or to the thickness  $a_\perp$  of the string). Therefore, one must impose the restriction  $\ell_n \gg l_v$  for the expressions as given by (3.11).

Using the above-mentioned relation between the effective potential strength  $g$  and the parameters of the square-well potential, the relation  $\ell_n = l_v$  leads to the crossover temperatures  $T_*^n$  which are given by

$$(1/T_*^n)^2 = (1/T_u^f)^2 [1 + 1/2c(N - n)n] \quad (3.13)$$

Up to  $T \simeq T_*^n$ , the mean separation  $\ell_n$  is comparable to the microscopic scale  $l_v$ ; for  $T \gtrsim T_*^n$ , this separation starts to grow and to become dominated by shape fluctuations. It follows from (3.13) that the crossover temperatures  $T_*^1 = T_*^{N-1}$  of the two outmost strings differ from the unbinding temperature  $T_u^f$  by a term of order  $1/N$ . Furthermore, the crossover temperature  $T_*^{N/2}$  for the separation of the inmost string pair differs from  $T_u^f$  by a term of order  $1/N^2$ .

This implies that the critical region in which the string behavior is dominated by fluctuations becomes very small for large  $N$ : the outmost strings start to fluctuate for  $T = T_*^1 = T_*^{N-1} \approx T_u^f(1 - 1/4cN)$  and the inmost strings start to become swollen for  $T = T_*^{N/2} \approx T_u^f(1 - 1/cN^2)$ . Therefore, for large  $N$ , the unbinding transition will resemble a discontinuous transition: the bunch will remain essentially unchanged up to  $T = T_*^1$  but will be completely unbound at the slightly higher temperature  $T = T_u^f$ .

*Strings with many-body interactions.* As described above, the fits to the numerical data lead to both  $N$ -dependent amplitudes and to  $N$ -dependent critical exponents over the accessible range of length scales whereas the Bethe Ansatz for  $N$  strings leads to  $N$ -dependent amplitudes but universal critical exponents. For the small values of  $N$  studied numerically, the  $N$ -dependence of the amplitudes gives rise to a prefactor of order one which does not seem to explain the different scaling behavior as observed numerically.

There is, however, a subtle difference between the total interaction potential as studied numerically and the one used in the Bethe Ansatz. The numerically studied potential contains only contributions from neighbouring pairs of strings. In contrast the total interaction potential  $\sum_{i>j} V(l_i - l_j)$  studied in the Bethe Ansatz solution contains contributions from all pairs. The difference corresponds to a 3-string interaction which must be added to  $\sum_{i>j} V(l_i - l_j)$  and which is effectively repulsive.

Three strings which experience repulsive 2-string interactions but attractive 3-string interactions can be described by the necklace model as mentioned above. If the 3-string interaction is so attractive that it can bind the strings together, the bundle will undergo an unbinding transition which belongs to subregime (C) of the intermediate fluctuation regime. This subregime is characterized by a probability distribution for the separation fields which can be explicitly calculated [27] and which decays as a power law for large separations [10]. This implies that the mean separation of the strings is discontinuous at the transition whereas higher order moments of the separation fields diverge in a continuous way. Likewise, the contact probability  $\mathcal{P}_3$  for 3-string collisions stays finite at the transition.

The interplay of 2-string and of  $n$ -string interactions with  $n > 2$  has been studied in a systematic way by field-theoretic renormalization. [20] This calculation predicts that

an *attractive* 3-string interaction represents a marginally relevant perturbation and leads to a different universality class for the unbinding transition which should belong to the universality class of the necklace model (for identical strings) and thus to the intermediate fluctuation regime. A *repulsive* 3-string interaction, on the other hand, is marginally irrelevant at the unbinding transition of the string bundle: it should not change the asymptotic critical behavior but should lead to large corrections to scaling on intermediate scales. A quantitative analysis of this crossover remains to be done.

From the analogy between strings and membranes, one expects that membranes with weak 2-membrane potentials and strongly attractive 3-membrane potentials should also undergo unbinding transitions which are characterized by a discontinuous behavior of the mean separations and by a finite contact probability  $\mathcal{P}_3$ . In real systems, attractive 3-membrane potentials can indeed arise from van der Waals forces.

#### 4. N membranes or strings at a rigid surface

As mentioned, many preparation methods lead to bunches of membranes which stick to a solid substrate or another surface. Thus, let us consider a bunch of  $N$  identical membranes with bending rigidity  $\kappa$  which adheres to a rigid surface.

##### 4.1. Repulsive hard wall interactions

It is again instructive to consider the case of the simple pair potential  $V_P(l) = V_{hw}(l) + Pl$  which consists of the steric hard wall potential and the pressure term. This corresponds to the situation in which the mutual interactions within the bunch are purely repulsive and the bunch is pushed against the rigid wall by the external pressure  $P$ .

Let us first focus on one pair of adjacent surfaces labelled by  $(n)$  and  $(n+1)$  with  $n = 0, 1, \dots, N-1$  which have the mean separation  $\ell_n$  and the bending rigidities  $\kappa_n$  and  $\kappa_{n+1}$ , respectively. If one ignores all other surfaces, the mean separation  $\ell_n$  is governed by the effective bending rigidity  $\kappa_{\text{eff}} = \kappa_n \kappa_{n+1} / (\kappa_n + \kappa_{n+1})$ .

For the bunch of  $N$  identical membranes with  $\kappa_n = \kappa$  adjacent to the rigid surface with  $\kappa_0 = \infty$ , these effective bending rigidities are given (i) by  $\kappa_{\text{eff}} = \kappa$  for the mean separation  $\ell_0$  of the first membrane from the rigid surface and (ii) by  $\kappa_{\text{eff}} = \kappa/2$  for the mean separations  $\ell_n$  with  $n \geq 1$ . If one uses these rigidities in the expression for the fluctuation-induced potential  $V_{fl}$  as given by (2.1), one expects the asymptotic relation  $\ell_0 \approx \ell_n/2^{1/3}$  with  $n \geq 1$  for small pressure  $P$ . This is exactly what is observed in the MC simulations. [12, 13]

*Contact probabilities.* The presence of the rigid surface breaks the translational invariance of the system. This should affect the contact probabilities  $\mathcal{P}_n^{(s)}$  for  $n$  manifolds to be close to the rigid surface as can be explicitly calculated for a bundle of  $n$  strings between two rigid surfaces at separation  $L_\perp$ . For this latter system, one finds from the free fermion

ground state that these contact probabilities have the scaling behavior

$$\mathcal{P}_n^{(s)} \sim 1/L_\perp^{2\zeta_n^{(s)}} \quad \text{with} \quad \zeta_n^{(s)} = (2n^2 + n)/2. \quad (4.1)$$

Comparison with (3.5) shows that  $\zeta_1^{(s)} = \zeta_2$  but that  $\zeta_n^{(s)} \neq \zeta_{n+1}$  for  $n > 2$ . This behavior should also apply to a system with only one rigid surface where the mean separation of the  $n$ 'th string from the rigid surface plays the role of  $L_\perp$ . One nontrivial test is provided by the necklace model for three strings if one considers the limit in which the line tension of one of the outer strings, say  $\sigma_1$ , becomes infinite. In this limit, one has  $\theta = \pi/4$  and thus  $\zeta_3(\pi/4) = 5$  as follows from (3.6). On the other hand, far from the rigid wall, one should recover the contact probabilities  $\mathcal{P}_n$  for a freely suspended bundle as given by (3.5).

Using the analogy between strings and membranes, one now predicts that the contact probabilities  $\mathcal{P}_n^{(s)}$  for  $n$  membranes to be close to a rigid surface scale as  $\mathcal{P}_n^{(s)} \sim 1/L_\perp^{\zeta_n^{(s)}}$  with  $\zeta_n^{(s)} = 2n^2 + n$ .

##### 4.2. Attractive interactions and unbinding transitions

Now, consider the case of direct interactions with a single potential minimum as shown in Fig. 2a. In general, the interaction  $V^{(s)}(l)$  between the rigid surface or wall and the first membrane will differ from the mutual interaction  $V(l)$  of two membranes.

*Possible unbinding temperatures.* As described in the previous section, an arbitrary number of freely suspended membranes unbinds at the unique unbinding temperature  $T = T_u^f$ . Therefore, a bunch containing more than one membrane cannot unbind from the substrate surface for  $T > T_u^f$ . In particular, the whole bundle can only unbind from the substrate for  $T \leq T_u^f$ .

On the other hand, for  $T < T_u^f$ , a single membrane cannot unbind from the remaining  $(N-1)$  membranes at the surface since this  $(N-1)$ -bunch must have an effective bending rigidity which is larger than the freely suspended  $(N-1)$ -bunch.

Thus, one concludes that there are only two possibilities: (i) The whole bunch unbinds from the wall for  $T = T_u^b \leq T_u^f$ , and (ii) single membranes peel off from the bunch for  $T > T_u^f$ . Depending on the relative strength of the surface-membrane and the membrane-membrane interaction, these two types of behavior are indeed found from explicit calculations.

*Numerical studies.* The unbinding of  $N = 1$  and 2 membranes from a rigid surface has been studied numerically by Monte Carlo simulations. The surface potential  $V^{(s)}(l)$  was taken to be a square well and to be identical with the mutual interaction potential  $V(l)$ . In this case, one observes a sequence of two unbinding transitions at  $T = T_u^2 > T_u^f$  and  $T = T_u^1 > T_u^2$  at which the outermost membrane unbinds from the surface. Likewise, the unbinding of  $N = 1, 2$  and 3 strings from a rigid line with  $V^{(s)}(l) = V(l)$  has been studied by numerical iterations of the transfer matrix. As a result,

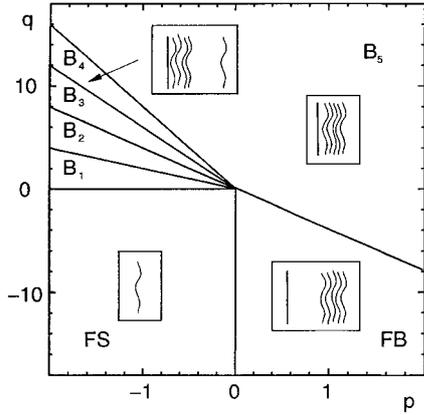


Fig. 3. Phase diagram for bundles of  $N = 5$  strings: the two parameters  $p \equiv -g$  and  $q \equiv -g^{(s)}$  denote the effective strength of the string-string and the surface-string interaction, respectively [4]

one finds a sequence of three unbinding transitions at three characteristic temperatures  $T_u^n$  with  $T_u^f < T_u^3 < T_u^2 < T_u^1$ . At all of these transitions, the diverging length scales are governed by the unbinding exponent  $\psi = 1$

In the continuum limit, the temperature  $T_u^1 = \sqrt{2} T_u^f$  since  $T_u \sim \sqrt{\kappa_{\text{eff}}}$  and the effective bending rigidity is given by  $\kappa_{\text{eff}} = \kappa/2$  for two identical membranes and by  $\kappa_{\text{eff}} = \kappa$  for one membrane at the rigid surface. In the numerical studies, this temperature window for the sequence of unbinding transitions is reduced by the presence of the small-scale cutoff (for the parameters as chosen in [3], one has  $T_u^1 = 1.14 T_u^f$ ).

*Strings with pair potentials.* The unbinding of  $N$  strings from a rigid line can be studied by the Bethe Ansatz. [4] The model now depends on the effective strength  $g$  of the pair potential for two strings and on the effective strength  $g^{(s)}$  of the interaction potential between the rigid surface and the adjacent string. The phase diagram of this model is shown in Fig. 3 for  $N = 5$

In regime ( $B_N$ ), all  $N$  strings are bound to the wall. For  $g < 0$ , i.e., for effectively attractive string-string interactions, all  $N$  strings unbind simultaneously from the wall at  $g^{(s)} = -(N-1)g$ . In regime (FB), the strings form a free (or unbound) bundle. For  $g > 0$ , on the other hand, the strings undergo a sequence of successive unbinding transitions. The  $n$ th string peels off from the rigid surface at  $g^{(s)} = -2(n-1)g$ , and one finds  $N-1$  different regimes ( $B_n$ ) with  $1 \leq n \leq N-1$  strings bound to the rigid surface. When the point  $g = g^{(s)} = 0$  is approached from regime ( $B_N$ ), the strings unbind simultaneously from the wall as well as from each other. For  $g^{(s)} > 0$  and  $g > 0$ , one has the free string (FS) regime.

Across all phase boundaries displayed in Fig. 3, both the free energy  $f^{(s)}$  and its first derivative  $\partial f^{(s)}/\partial g^{(s)}$  are continuous whereas its second derivative with respect to  $g^{(s)}$  exhibits a discontinuity. Hence these transitions are of second order with the critical exponent  $\alpha = 0$  for the specific heat. At the phase boundary between ( $B_N$ ) and ( $B_{N-1}$ ), the mean separations  $\ell_n$  with  $n < N-1$  are continuous but their

first derivatives  $\partial \ell_n / \partial g^{(s)}$  exhibit a jump. The critical exponent for the diverging length scales has the universal value  $\psi = 1$  at each transition line, even at the point  $g = g^{(s)} = 0$

Regime ( $B_N$ ) and regime (FS) are attained at sufficiently low and sufficiently high temperatures, respectively. Depending on the relative strength of the string-string and the surface-string attraction, the temperature trajectory will move from ( $B_N$ ) to (FS) via the free bundle regime (FB) or via the intermediate states ( $B_n$ ). In the latter case one has a sequence of  $N$  unbinding transitions. In the limit of infinite  $N$ , the corresponding sequence of critical temperatures  $T_u^n$  attains the finite value  $T_u^f$  [3]. It now follows from the explicit expressions for the phase boundaries that [4]

$$1 - (T_u^f/T_u^n)^2 \sim 1/n^\lambda \quad \text{with} \quad \lambda = 1 \quad . \quad (4.2)$$

## 5. Summary and outlook

In summary, the two systems shown in Fig. 1 exhibit different types of unbinding transitions:

(i)  $N$  freely suspended membranes or strings unbind at a unique unbinding temperature which is independent of  $N$ . The corresponding density profiles strongly depend on  $N$ , however, compare the expression (3.11) for the mean separations within the bunch. Therefore, in the limit of large  $N$ , the transition will be rather abrupt and thus will resemble a discontinuous transition even though it is still a continuous one; and

(ii) If the manifolds are attracted towards a rigid surface, relatively weak adhesion towards the surface leads to two unbinding transitions in which the whole bunch unbinds from the wall at  $T = T_u^b < T_u^f$  and then disintegrates into single membranes at  $T = T_u^f$ . Relatively strong adhesion, on the other hand, leads to a sequence of  $N$  successive unbinding transitions in which one membrane after another peels off from the bunch. Thus, the presence of the rigid surface is felt by the outmost membrane even if there is a large number of intervening membranes.

In all cases discussed above, the unbinding behavior of (fluid) membranes and strings is found to be very similar. This analogy breaks down, however, as soon as the direct interaction exhibits a potential barrier as shown in Fig. 2b. If the barrier decays faster than  $\sim 1/l^2$  for large  $l$ , strings in two dimensions always tunnel through such a barrier and thus always undergo continuous unbinding transitions. Even though membranes tunnel through sufficiently weak barriers, they are trapped by sufficiently large ones. Thus, the unbinding transition of membranes through a potential barrier is continuous and discontinuous for weak and for strong barriers, respectively. This follows from simple stability arguments in which one considers the free energy of island excitations through the potential barrier [35], and is consistent with recent Monte Carlo simulations [36].

It is interesting to note that the qualitative features discussed here for fluid membranes should also apply to bunches of polymerized membranes. For example, it would be interesting to determine the contact probabilities within a bunch of polymerized membranes with hard-wall interactions. This would give another non-trivial check for the scaling relation (3.7) which expresses the contact exponent  $\zeta_2$  in terms of the roughness exponent  $\zeta$ .

## Appendix A. Renormalized interactions

Consider a bunch of  $N$  membranes labelled by  $(n)$  with  $n = 1, 2, \dots, N$  which have bending rigidities  $\kappa_n$  and which are, on average, parallel to a reference plane. The distance of membrane  $(n)$  from this reference plane is denoted by the height variable  $h_n$ . The two adjacent membranes labelled by  $(n)$  and  $(n+1)$  interact via the pair potential  $V_n(h_{n+1} - h_n)$  and are pushed together by external pressures  $P_n$  with  $n = 1, 2, \dots, N-1$ . In a real system, these pressures are usually equal to the overall pressure  $P$  acting on the whole bunch, but it is conceptually useful to allow these pressures to vary within the bunch.

The conformations of the bunch is then governed by the configurational energy or effective Hamiltonian as given by

$$\mathcal{H}\{\mathbf{h}|\mathbf{P}\} = \int d^2x \left\{ \sum_{n=1}^{N-1} [V_n(h_{n+1} - h_n) + P_n(h_{n+1} - h_n)] + \frac{1}{2} \sum_{n=1}^N \kappa_n (\nabla^2 h_n)^2 \right\} \quad (\text{A.1})$$

For  $N$  strings with line tensions  $\sigma_n$ , the effective Hamiltonian has essentially the same form: for strings, the internal coordinate  $x$  is 1-dimensional, and the bending terms  $\kappa_n (\nabla^2 h_n)^2$  must be replaced by the tension terms  $\sigma_n (dh_n/dx)^2$

In the absence of any additional constraint, the system is translationally invariant in the perpendicular direction, and the translation of the whole bunch represents a 'zero mode'. As a consequence, the 'center-of-mass' coordinate  $h_0$  decouples from the  $N-1$  separation fields  $l_n \equiv h_{n+1} - h_n$ , and the effective Hamiltonian can be divided as  $\mathcal{H}\{\mathbf{h}|\mathbf{P}\} = \mathcal{H}_0\{h_0\} + \mathcal{H}'\{\mathbf{l}|\mathbf{P}\}$ . The partition function is obtained by a summation over the separation fields  $l_n$ , and the corresponding free energy per unit area is given by

$$G(\mathbf{P}) = -T \ln \left[ \int \prod_{n=1}^{N-1} d\{l_n\} \exp[-\mathcal{H}'\{\mathbf{l}|\mathbf{P}\}/T] \right] / \mathcal{A} \quad (\text{A.2})$$

where  $\prod d\{l_n\}$  represents the multiple path integral over all configurations of  $\{l_1, l_2, \dots, l_{N-1}\}$  and  $\mathcal{A}$  denotes the area of the manifolds when projected onto the reference plane.

One may now perform a Legendre transformation from the pressure variables  $P_n$  to the separation variables  $\ell_n \equiv \langle l_n \rangle = \langle h_{n+1} - h_n \rangle = \partial G / \partial P_n$ , and define the Legendre-transformed free energy per unit area via

$$F(\ell) \equiv G(\mathbf{P}(\ell)) - \sum_{n=1}^{N-1} P_n \ell_n \quad (\text{A.3})$$

This free energy density contains the renormalized interactions of all nearest neighbor pairs of manifolds within the bunch since  $-\partial F(\ell) / \partial \ell_n = P_n$

*Constant pressure.* If all pressures  $P_n$  are equal to the overall external pressure  $P$ , one has  $\partial G / \partial P = \langle h_N - h_1 \rangle \equiv \ell_{1N}$  i.e., the pressure is now conjugate to the mean thickness of the whole bunch or bundle. The Legendre-transformed free energy per unit area is then given by

$$F(\ell_{1N}) \equiv G(P(\ell_{1N})) - P \ell_{1N} \quad \text{with} \quad -\partial F(\ell_{1N}) / \partial \ell_{1N} = P \quad (\text{A.4})$$

Thus, this free energy density can be viewed as an effective interaction for the whole bunch, see Sect. 3.1.

*Two manifolds.* For  $N=2$ , the mean separation of the two manifolds is denoted by  $\ell_{12} \equiv \ell$ , and the corresponding free energy density  $F(\ell_{12}) = F(\ell)$  represents the renormalized interaction of these two manifolds. This definition of the renormalized interaction holds for any direct interaction  $V(l)$ . As usual, if the system undergoes a phase transition, the equation of state as given by  $\partial G / \partial P = \ell$  has several solutions for  $\ell$  corresponding to several distinct states of the system.

## Appendix B. Renormalization of steric hard wall potentials

In this appendix, systematic procedures for the derivation of the fluctuation-induced interaction will be discussed. Such a derivation must start from the steric hard wall interaction  $V_{hw}(l)$  as defined by

$$V_{hw}(l) \equiv \begin{cases} \infty & \text{for } l < 0 \\ 0 & \text{for } l > 0 \end{cases} \quad (\text{B.1})$$

which incorporates the constraint that the manifolds cannot intersect one another. This potential represents the simplest direct interaction since it contains no energy and no length scale.

The fluctuation-induced interaction  $V_{fl}$  can be viewed as the effective interaction which arises from the renormalization of the hard wall interaction  $V_{hw}(l)$ . It turns out, however, that there are several possible ensembles which, in general, lead to somewhat different values for the dimensionless coefficient  $c_{fl}$  which governs the strength of  $V_{fl}$  compare (2.1)

*Pressure ensemble.* First, the manifolds which interact by repulsive hard wall potentials as in (B.1) can be kept together by the external pressure  $P$ . As explained in Appendix A, this pressure is conjugate to the separation  $\ell_{1N}$  of the two outer manifolds of the bunch, and the Legendre-transformed free energy is then given by  $F(\ell_{1N}) = G(P(\ell_{1N})) - P \ell_{1N}$  with  $-\partial F(\ell_{1N}) / \partial \ell_{1N} = P$ . Let us now anticipate that the mean separation  $\ell_n$  of two adjacent manifolds labelled by  $(n)$  and  $(n+1)$  is of the order of  $\bar{\ell} \equiv \ell_{1N} / (N-1)$  for all  $n$ . One may then define the free energy density per separation field

$$f_{hw}(\bar{\ell}) \equiv F(\ell_{1N}) = (N-1)\bar{\ell} / (N-1) \quad (\text{B.2})$$

which determines  $\bar{\ell}$  according to  $-\partial f_{hw}(\bar{\ell}) / \partial \bar{\ell} = P$ . Therefore, the quantity  $f_{hw}$  represents the effective interaction between two adjacent membranes as obtained from the renormalization of the hard wall interaction  $V_{hw}(l)$ , and may be identified with the fluctuation-induced interaction, i.e.,  $V_{fl}(\bar{\ell}) = f_{hw}(\bar{\ell})$

*Periodic slab.*  $N$  manifolds which interact via repulsive hard wall interactions may also be confined by external boundaries. From the theoretical point of view, it is often convenient to confine them by a slab (or strip) of width  $L_{\perp}$  and to impose periodic boundary conditions in the perpendicular direction (in two dimensions, this corresponds to the surface of a cylinder). In this geometry, the bunch or bundle is still translationally invariant in the perpendicular direction, which implies that all separations  $\ell_n$  between nearest neighbor manifolds are identical, i.e.,  $\ell_n = \bar{\ell} \equiv \ell$  and that the width of the strip is  $L_{\perp} = N\ell$ . The translational invariance also implies that the 'center-of-mass' coordinate again represents a 'zero mode' which undergoes free shape fluctuations (along the surface of the cylinder). However, all  $N$  fluctuating fields  $h_n$  give a finite contribution to the partition function (for finite  $L_{\perp}$ ), and one does not have to treat the 'zero mode' separately.

The partition function  $\mathcal{Z}'(L_{\perp})$  is now given by a multiple path integral over all configurations of  $\{h_1, h_2, \dots, h_N\}$ . The corresponding free energy density per manifold is defined via

$$f'_{hw}(\ell) \equiv -T \ln[\mathcal{Z}'(L_{\perp} = N\ell)] / \mathcal{A}N \quad (\text{B.3})$$

where  $\mathcal{A}$  is the projected area of the manifolds as before. This free energy density  $f'_{hw}(\ell)$  represents a second possible definition of the fluctuation-induced interaction  $V'_{fl}(\ell)$  which will, in general, differ from the Legendre-transformed free energy density  $f_{hw}(\bar{\ell})$

*Fixed slab.* As a third possibility, the  $N$  manifolds which repel one another via hard wall interactions may be confined by two rigid walls, i.e., by a slab (or strip) with fixed boundary conditions. In this geometry, the translational invariance is broken and the 'zero mode' is suppressed. There are  $N + 1$  local separation fields with mean values which depend, to some extent, on the position of the nearest neighbor pairs within the bunch or bundle. If the separation of the two rigid walls is denoted by  $L_{\perp}$ , these mean values are of the order of  $\bar{\ell} \equiv L_{\perp} / (N + 1)$

The height variable  $h_n$  now measures the separation of manifold ( $n$ ) from one of the two rigid walls, say the bottom one. The partition function  $\mathcal{Z}''$  is given by a multiple path integral over all configurations of  $\{l_1, l_2, \dots, l_N\}$  with  $l_1 \equiv h_1$  and  $l_n = h_{n+1} - h_n$  for  $n \geq 2$  as before, and the corresponding free energy density per manifold is

$$f''_{hw}(\bar{\ell}) \equiv -T \ln[\mathcal{Z}''(L_{\perp} = (N + 1)\bar{\ell})] / \mathcal{A}N \quad (\text{B.4})$$

This free energy density gives yet another definition of the fluctuation-induced interaction.

*Large rigidity limit.* In order to see if the different ensembles introduced in the previous subsections are equivalent, it is quite instructive to consider the limit of large rigidities in which all shape fluctuations are suppressed and in which each manifold reduces to a 'point particle', i.e., to a single degree of freedom. In this limit, the path integrals reduce to normal integrals, and the free energies defined in (B.2), (B.3), and (B.4) may be easily calculated.

In the pressure ensemble, one finds that the mean separation of two adjacent 'point particles' is independent of the position ( $n$ ), and the separation of two adjacent particles is given by  $\ell_n = \ell \equiv \ell_{1N} / (N - 1)$ . The free energy density as defined by (B.2) then becomes

$$f_{hw}(\ell) = T(\ln[a_{\perp}/\ell] - 1) \quad (\text{B.5})$$

with the small-scale cutoff  $a_{\perp}$ .

On the other hand, the partition functions  $\mathcal{Z}'$  and  $\mathcal{Z}''$  for the periodic and for the fixed slab are given by  $\mathcal{Z}' = (L_{\perp}/a_{\perp})^N / (N - 1)!$  and by  $\mathcal{Z}'' = (L_{\perp}/a_{\perp})^N / N!$ , respectively. For finite  $N$ , the corresponding free energy densities  $f'_{hw}(\ell)$  and  $f''_{hw}(\ell)$  differ from  $f_{hw}(\ell)$  as given by the expression (B.5) but they become identical with this expression in the limit of large  $N$ .

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