

Dynamic pattern evolution on scale-free networks

Haijun Zhou and Reinhard Lipowsky[†]

Max Planck Institute of Colloids and Interfaces, 14424 Potsdam, Germany

Edited by Michael E. Fisher, University of Maryland, College Park, MD, and approved May 25, 2005 (received for review December 14, 2004)

A general class of dynamic models on scale-free networks is studied by analytical methods and computer simulations. Each network consists of N vertices and is characterized by its degree distribution, $P(k)$, which represents the probability that a randomly chosen vertex is connected to k nearest neighbors. Each vertex can attain two internal states described by binary variables or Ising-like spins that evolve in time according to local majority rules. Scale-free networks, for which the degree distribution has a power law tail $P(k) \sim k^{-\gamma}$, are shown to exhibit qualitatively different dynamic behavior for $\gamma < 5/2$ and $\gamma > 5/2$, shedding light on the empirical observation that many real-world networks are scale-free with $2 < \gamma < 5/2$. For $2 < \gamma < 5/2$, strongly disordered patterns decay within a finite decay time even in the limit of infinite networks. For $\gamma > 5/2$, on the other hand, this decay time diverges as $\ln(N)$ with the network size N . An analogous distinction is found for a variety of more complex models including Hopfield models for associative memory networks. In the latter case, the storage capacity is found, within mean field theory, to be independent of N in the limit of large N for $\gamma > 5/2$ but to grow as N^α with $\alpha = (5 - 2\gamma)/(\gamma - 1)$ for $2 < \gamma < 5/2$.

random network | Boolean dynamics | cellular automata | associative memory

The biosphere contains many complex networks built up from rather different elements such as molecules, cells, organisms, or machines. Despite their diversity, these networks exhibit some universal features and generic properties, a topic of much recent interest (1–6). One important result of this recent activity is a classification scheme for the structure of networks in terms of their topology and connectivity (5, 6). The basic elements of each network can be represented by nodes or vertices and their interrelations by edges between these vertices. By definition, the degree, k , of a given vertex is equal to the number of edges connected to it. In this way, each network is characterized by its graph, a well defined mathematical object (7). Four large subsets of network graphs have been characterized in terms of their connectivity properties: regular ordered and random networks, which have been studied for a long time, as well as small-world (1) and scale-free (4) networks. The latter types of networks are characterized by a degree distribution, $P(k)$, that decays as $P(k) \sim 1/k^\gamma$ with decay exponent γ .

Many biological, social, and technological networks are found to be scale-free (5, 6). To explain this abundance, several mechanisms have been proposed, some of which are related to the growth of networks with preferential attachment rules (5, 8). The impact of network architecture on network integrity or resilience has also been investigated (6, 9–11). Upon random removal of a finite fraction of vertices, a scale-free network with decay exponent $2 < \gamma < 3$ will always keep a giant component. In contrast, if a tiny fraction of the most highly connected vertices is selectively removed, such a network will also break up into many small components. Applying this insight to the case of disease spreading on networks, one realizes that infective diseases have a high probability to affect the whole network (12) unless one immunizes the most highly connected vertices (13).

Random removal of vertices is equivalent to the process of site percolation on the initial network (9, 10). Likewise, disease spreading is intimately related, in the long time limit, to bond

percolation (6, 10). In general, the elements of real networks are dynamic and exhibit various properties that change with time. A more detailed description of the network dynamics is then obtained in terms of dynamical variables associated with each vertex of the network. We will consider processes on networks that evolve fast compared with any changes in the network topology, which is therefore taken to be time-independent. Two examples for such dynamical processes are provided by the regulation of genetic networks that exhibit a changing pattern of active and inactive genes (see, e.g., ref. 14) or by neural networks that can be characterized by firing and nonfiring neurons (see, e.g., ref. 15).

To identify universal features and generic properties, it is often convenient to use discrete dynamical variables and to allow each vertex to attain several distinct states. The network is then characterized, at each moment in time, by a certain pattern of these internal vertex states, and the time evolution of these patterns represents the global dynamics of the whole network.

The study of discrete dynamical processes on ordered and random networks has a long history. Ordered networks have been studied in the context of neural computation (16, 17), cellular automata or Boolean dynamics (18, 19), and dynamic Ising models (20). Random networks with random Boolean dynamics were introduced in the context of gene expression and studied as prototypes of random cellular automata (21–23). Discrete dynamic processes on small-world networks were discussed in ref. 2, whereas analogous processes on scale-free networks have only been addressed rather recently (24–26).

Random networks as originally studied in mathematics (5–7) are characterized by a Poissonian degree distribution as given by $P(k) = \langle k \rangle^k e^{-\langle k \rangle} / k!$, where $\langle k \rangle \equiv \sum k P(k)$ denotes the mean value of the vertex degree k . In contrast, scale-free networks are characterized by $P(k) \sim k^{-\gamma}$ with decay exponent γ for an intermediate range of k -values that can be defined as follows. To avoid disconnected subgraphs, it is useful to introduce a certain minimal degree $k_0 > 0$. Furthermore, a large but finite network containing N vertices is also characterized by a certain maximal degree k_N . These two “cutoffs” can be incorporated in a particularly simple way via the explicit form (9)

$$P(k) \equiv \begin{cases} (1/\mathcal{A})k^{-\gamma} & \text{for } k_0 \leq k \leq k_N \\ 0 & \text{otherwise} \end{cases} \quad [1]$$

with the normalization factor $\mathcal{A} \equiv \sum P(k)$ and $k_N = k_0 N^{1/(\gamma - 1)}$ (see Degree Distribution and Maximal Vertex Degree in Supporting Text, which is published as supporting information on the PNAS web site).

Many scale-free networks are characterized by decay exponents γ that fall into the narrow range $2 < \gamma \leq 5/2$ as observed in ref. 24. Indeed, table 2 of ref. 5 contains a list of ten scale-free networks with $2 < \gamma < 5/2$, one with $\gamma = 2.5$, and only three with $2.5 < \gamma < 3$. The lower boundary value $\gamma = 2$ is easy to understand since it ensures that the mean vertex degree $\langle k \rangle = \sum k P(k)$ remains finite in the limit of large N . The upper boundary

This paper was submitted directly (Track II) to the PNAS office.

Freely available online through the PNAS open access option.

[†]To whom correspondence should be addressed. E-mail: lipowsky@mpikg.mpg.de.

© 2005 by The National Academy of Sciences of the USA

with $0 \leq Q(t) \leq 1$. Note that the ordering probability Q differs, in general, from the overall probability to find any vertex in the spin-up state. The latter probability is given by $\langle q \rangle \equiv \sum_k P(k)q_k$. Special patterns for which $\langle q \rangle = Q$ are provided by k -independent probabilities $q_k = q$. In particular, for the all-spin-down pattern with $q_k = 0$ for all k and for the all-spin-up pattern with $q_k = 1$ for all k , one has $Q = \langle q \rangle = 0$ and $Q = \langle q \rangle = 1$, respectively.

If we know the ordering probability Q at a certain time t , we can calculate the probabilities q_k at the next time step. Indeed, it follows from the majority rule dynamics as defined by Eq. 2 that

$$q_k(t+1) = \sum'_m \left(1 - \frac{1}{2} \delta_{m,k/2}\right) B_{k,m} Q^m(t) (1-Q(t))^{k-m}, \quad [4]$$

where the prime at the summation symbol indicates that this sum runs over all integer m with $k/2 \leq m \leq k$, δ is the Kronecker symbol, and $B_{k,m} \equiv k!/[m!(k-m)!]$ are the binomial coefficients. Finally, summation of the left-hand side of Eq. 4 leads to the evolution equation

$$Q(t+1) = \Psi(Q(t)) \quad [5]$$

for the ordering probability Q with the evolution function

$$\Psi(Q) \equiv \sum_k \sum'_m \left(1 - \frac{1}{2} \delta_{m,k/2}\right) k P(k) B_{k,m} Q^m (1-Q)^{k-m} / \langle k \rangle \quad [6]$$

for random graphs with no vertex degree correlations as considered here.

The evolution equation (5) has two stable fixed points at $Q = 0$ and $Q = 1$, and an unstable one at $Q = 1/2$. The fixed points with $Q = 0$ and $Q = 1$ correspond to the all-spin-down and all-spin-up patterns, respectively. The unstable fixed point with $Q = 1/2$ represents the phase boundary between these two ordered patterns; the corresponding boundary patterns are characterized by probabilities \hat{q}_k , which satisfy

$$\hat{Q} \equiv \sum_k k P(k) \hat{q}_k / \langle k \rangle = 1/2 \quad \text{or} \quad N \langle k \rangle \hat{Q} = M, \quad [7]$$

where M is the total number of edges. A subset of these boundary patterns is provided by those patterns for which the probabilities $\hat{q}_k = 1/2$ for all vertex degrees k . Therefore, the order parameter of these systems is taken to be

$$y \equiv Q - \hat{Q} = Q - 1/2, \quad [8]$$

which vanishes for all boundary patterns.

In the next section, we will compare the mean field trajectories obtained from the evolution equation (5) with the pattern evolution obtained from computer simulations. In the latter case, one starts from a certain initial pattern and applies the majority rule dynamics as given by Eq. 2 to each vertex of the network. Since we want to be able to use the same initial pattern for both mean field theory and computer simulations, we will define the enlarged set of *strongly disordered patterns* that consists (i) of the boundary patterns characterized by Eq. 7 and (ii) of additional patterns characterized by probabilities \tilde{q}_k that lead to $Q = \tilde{Q} = 1/2 \pm 1/2M$ and, thus, to the order parameter

$$\tilde{y} \equiv \sum_k k P(k) \tilde{q}_k / \langle k \rangle - 1/2 = \pm 1/2M = \pm 1 / \langle k \rangle N, \quad [9]$$

which vanishes in the limit of large network size N . A pattern with probabilities \tilde{q}_k can be obtained from a boundary pattern by simultaneously flipping two spins with opposite orientations on a k and $(k+1)$ vertex.

The evolution equation for the order parameter $y = Q - 1/2$ can be directly obtained from the corresponding equation (7) for Q and is given by $y(t+1) = \Psi(1/2 + y(t)) - 1/2$. In the vicinity of the unstable fixed point with $Q = 1/2$ and $\Psi(1/2) = 1/2$, the latter equation can be linearized and becomes

$$y(t+1) \approx \Psi'(1/2)y(t) \quad \text{with} \quad \Psi'(1/2) \equiv d\Psi/dQ|_{Q=1/2} \quad [10]$$

for small $y(t)$. Iterating this equation n times from an initial time t_0 up to a final time t_1 , one obtains the time difference

$$\Delta t_{01} \equiv t_1 - t_0 \approx \frac{\ln|y(t_1)| - \ln|y(t_0)|}{\ln \Psi'(1/2)} \quad [11]$$

in the limit of small $y(t_0)$.

It follows from the explicit expression (Eq. 6) for the evolution function $\Psi(Q)$ that the derivative $\Psi'(1/2)$ at the unstable fixed point with $Q = 1/2$ diverges for a scale-free network with decay exponent $\gamma \leq 5/2$ in the limit of large network size N . More precisely, one obtains the asymptotic behavior

$$\begin{aligned} \Psi'(1/2) &\approx c_\gamma k_0^{1/2} && \text{for } \gamma > 5/2 \\ &\approx c_\gamma k_0^{1/2} \ln(N) && \text{for } \gamma = 5/2 \\ &\approx c_\gamma k_0^{1/2} N^{\alpha/2} && \text{for } 2 < \gamma < 5/2 \end{aligned} \quad [12]$$

in the limit of large N with $\alpha = (5 - 2\gamma)/(\gamma - 1)$, where c_γ is a dimensionless γ -dependent coefficient (see *Derivation of Eq. 12 in Supporting Text*).

Results and Discussion

Decay of Strongly Disordered Patterns. Now, consider an initial state of the network at time $t = t_0$ that corresponds to a strongly disordered pattern with order parameter $y(t_0) = \tilde{y} = \pm 1/2M \sim 1/N$ as in Eq. 9. For $t > t_0$, such an initial pattern evolves according to the majority rule dynamics that leads to an increase in the absolute value of the order parameter y and, thus, to the growth of order. We will characterize this decay of the strongly disordered patterns by the decay time t_d , which is the time it takes to reach a pattern with an order parameter y_* that satisfies $|y_*| \geq 1/4$. In addition, we have also estimated the probability to reach the completely ordered patterns when one initially starts from strongly disordered ones.

Within mean field theory, the decay time t_d follows from Eq. 11 and is given by $t_d \approx \ln(\langle k \rangle N) / \ln(\Psi'(1/2))$ for large $\langle k \rangle$ or large N . This expression behaves as

$$\begin{aligned} t_d &\sim \ln(N) && \text{for } \gamma > 5/2 \\ &\sim \ln(N) / \ln \ln(N) && \text{for } \gamma = 5/2 \\ &\sim 2(\gamma - 1) / (5 - 2\gamma) && \text{for } 2 < \gamma < 5/2 \end{aligned} \quad [13]$$

in the limit of large network size N . Thus, for scale-free networks with decay exponent $2 < \gamma < 5/2$, strongly disordered patterns always decay after a *finite* number of iteration steps even in the limit of large N . In contrast, for networks with $\gamma > 5/2$, the decay time diverges as $\ln(N)$. The latter behavior for $\gamma > 5/2$ also applies to Poissonian networks and is, thus, consistent with recent results for opinion spreading on social networks (28).

We have confirmed these mean field predictions by computer simulations. To avoid both disconnected and tree-like network graphs, the simulations were performed for networks with a relatively large mean vertex degree $\langle k \rangle$ and minimal vertex degree $k_0 > 1$. In Fig. 1, we display the time evolution of the absolute value of the order parameter, $|y| = |Q - 1/2|$, as

Summary. In summary, we have studied the response behavior of a large class of dynamic network models. We found that scale-free networks are characterized by very rapid response provided $\gamma < 5/2$. As far as the decay time of strongly disordered patterns is concerned, $\gamma = 5/2$ represents a sharp borderline. For $\gamma < 5/2$, all of these patterns decay within a finite decay time even in the limit of infinite vertex number. For $\gamma > 5/2$, on the other hand, this time diverges as $\ln(N)$ for large vertex number N . We also determined the minimal fraction of spin flips to induce a transition from one ordered pattern to another ordered pattern. This fraction becomes very small as one reduces the value of γ and vanishes as an essential singularity in $\gamma - 2$. These results are generic and can be generalized to more complex models that involve more than two vertex states and/or directed edges.

One particularly interesting extension is to Hopfield models for associative memory on scale-free networks. In this latter case, we find that the storage capacity is independent of network size N for $\gamma > 5/2$ but grows as N^α with $\alpha = (5 - 2\gamma)/(\gamma - 1)$ for $2 < \gamma < 5/2$. For the functional networks of the human brain with $\gamma \approx 2.1$ (32), this implies that the storage capacity grows as N^α with $\alpha \approx 0.73$. These mean field predictions should be

accessible to simulation studies that have been restricted, so far, to the special case $\gamma = 3 > 5/2$ (35).

Our results shed light on the empirical observation that many real-world networks are scale-free with decay exponent $2 < \gamma < 5/2$. Further extensions of our study should address possible effects of vertex-vertex correlations, which are present in real-world networks and lead to clustering (1) and modularity (3, 36). We expect that correlations between vertices of high degree will make the response of scale-free networks with $\gamma < 5/2$ even more rapid, but this remains to be studied in more detail. Likewise, it will be interesting to investigate how the response behavior is influenced by weighted edges (see, e.g., ref. 37) corresponding to variable nearest neighbor couplings.

Note Added in Proof. For the special case of scale-free networks with minimal vertex degree $k_0 = 1$, additional simulations have confirmed that the decay of strongly disordered patterns is characterized by persistent oscillations of the order parameter y around $y = 0$.

We thank Max Aldana for interesting correspondence about his previous work on scale-free networks.

1. Watts, D. J. & Strogatz, S. H. (1998) *Nature* **393**, 440–442.
2. Watts, D. J. (1999) *Small Worlds* (Princeton Univ. Press, Princeton).
3. Hartwell, L. H., Hopfield, J. J., Leibler, S. & Murray, A. W. (1999) *Nature* **402**, C47–C52.
4. Barabási, A.-L. & Albert, R. (1999) *Science* **286**, 509–512.
5. Albert, R. & Barabási, A.-L. (2002) *Rev. Mod. Phys.* **74**, 47–97.
6. Newman, M. E. J. (2003) *SIAM Rev.* **45**, 167–256.
7. Bollobas, B. (1998) *Modern Graph Theory* (Springer, New York).
8. Dorogovtsev, S. N. & Mendes, J. F. F. (2002) *Adv. Phys.* **51**, 1079–1187.
9. Cohen, R., Erez, K., Ben-Avraham, D. & Havlin, S. (2000) *Phys. Rev. Lett.* **85**, 4626–4628.
10. Callaway, D. S., Newman, M. E. J., Strogatz, S. H. & Watts, D. J. (2000) *Phys. Rev. Lett.* **85**, 5468–5471.
11. Valente, A. X., Sarkar, A. & Stone, H. A. (2004) *Phys. Rev. Lett.* **92**, 118702.
12. Boguna, M., Pastor-Satorras, R. & Vespignani, A. (2003) *Phys. Rev. Lett.* **90**, 028701.
13. Madar, N., Kalisky, T., Cohen, R., Ben-Avraham, D. & Havlin, S. (2004) *Eur. Phys. J. B* **38**, 269–276.
14. Tong, A. H. Y., Lesage, G., Bader, G. D., Ding, H., Xu, H., Xin, X., Young, J., Berriz, G. F., Brost, R. L., Chang, M., et al. (2004) *Science* **303**, 808–813.
15. Dayan, P. & Abbott, L. (2001) *Theoretical Neuroscience* (MIT Press, Cambridge, MA).
16. McCulloch, W. S. & Pitts, W. (1943) *Bull. Math. Biophys.* **5**, 115–133.
17. Hopfield, J. J. (1982) *Proc. Natl. Acad. Sci. USA* **79**, 2554–2558.
18. von Neumann, J. (1963) in *John von Neumann, Collected Works*, ed. Taub, A. H. (Pergamon, Oxford), Vol. 5.
19. Wolfram, S. (1983) *Rev. Mod. Phys.* **55**, 601–644.
20. Glauber, R. J. (1963) *J. Math. Phys.* **4**, 294–307.
21. Kauffman, S. A. (1969) *J. Theor. Biol.* **22**, 437–467.
22. Derrida, B. & Pomeau, Y. (1986) *Europhys. Lett.* **1**, 45–49.
23. Aldana, M., Coppersmith, S. & Kadanoff, L. P. (2003) *Perspectives and Problems in Nonlinear Science*, eds. Marsden, J. E., Kaplan, E. & Sreenivasan, K. R. (Springer, New York).
24. Aldana, M. & Cluzel, P. (2003) *Proc. Natl. Acad. Sci. USA* **100**, 8710–8714.
25. Bar-Yam, Y. & Epstein, I. R. (2004) *Proc. Natl. Acad. Sci. USA* **101**, 4341–4345.
26. Li, F., Long, T., Lu, Y., Ouyang, Q. & Tang, C. (2004) *Proc. Natl. Acad. Sci. USA* **101**, 4781–4786.
27. Derrida, B., Gardner, E. & Zippelius, A. (1987) *Europhys. Lett.* **4**, 167–173.
28. Krapivsky, P. L. & Redner, S. (2003) *Phys. Rev. Lett.* **90**, 238701.
29. Mélin, R., Anglès d'Auriac, J. C., Chandra, P. & Douçot, B. (1996) *J. Phys. A* **29**, 5773–5804.
30. Spirin, V., Krapivsky, P. L. & Redner, S. (2001) *Phys. Rev. E* **65**, 016119.
31. Svenson, P. (2002) *Phys. Rev. E* **64**, 036122.
32. Eguiluz, V. M., Chialvo, D. R., Cecchi, G. A., Baliki, M. & Apkarian, A. V. (2005) *Phys. Rev. Lett.* **94**, 018102.
33. Häggström, O. (2002) *Physica A* **310**, 275–284.
34. Chialvo, D. R. (2004) *Physica A* **340**, 756–765.
35. Stauffer, D., Aharony, A., da Fontoura Costa, L. & Adler, J. (2003) *Eur. Phys. J. B* **32**, 395–399.
36. Maslov, S. & Sneppen, K. (2002) *Science* **296**, 910–913.
37. Zhou, H. & Lipowsky, R. (2004) *Lecture Notes Comput. Sci.* **3038**, 1062–1069.