Dynamic Pattern Evolution on Scale-free Networks

– Supporting Information –

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This Supporting Information contains six appendices A – F with some technical details of our calculations. Appendix A reviews the parametrization of the scale-free degree distribution $P(k)$ and the behavior of the maximal vertex degree k_N. The configuration model used to generate the random scale-free networks numerically is briefly described in Appendix B. In Appendix C, we determine the first derivative of the evolution function $\Psi(Q)$ at the unstable fixed point with $Q = 1/2$ and, thus, derive equation (12) of the main text. Likewise, Appendix D contains the derivation of equation (14) of the main text. In Appendix E, we determine the storage capacity of Hopfield models on scale-free networks. Finally, we briefly discuss the extension of our mean field theory to directed scale-free networks in Appendix F.

The Supporting Figures 5 – 10 are attached at the end of this document.

A. Degree distribution and maximal vertex degree

Consider a scale-free network with N vertices i characterized by vertex degrees k_i. As explained in the introductory part of the main text, the degrees k_i are taken to satisfy $k_0 \leq k_i \leq k_N$ where k_0 and k_N denote the minimal and maximal vertex degree, respectively. The degree distribution $P(k)$ is taken to have the explicit form

$$\begin{aligned} P(k) &\equiv (1/\mathcal{A})k^{-\gamma} \quad \text{for } k_0 \leq k \leq k_N \\ &\equiv 0 \qquad\qquad\quad \text{otherwise} \end{aligned} \tag{A.1}$$

with the normalization factor $\mathcal{A} \equiv \sum P(k)$. Equation (A.1) is identical with equation (1) of the main text and is repeated here for convenience.

In the limit of large N, the degree distribution is normalizable for $\gamma > 1$. The normalization factor $\mathcal{A}$ is then given by

$$\mathcal{A} \equiv \sum_{k=k_0}^{k_N} k^{-\gamma} \approx \frac{k_0^{1-\gamma} - k_N^{1-\gamma}}{\gamma - 1} \quad . \tag{A.2}$$

The mean vertex degree is finite for $\gamma > 2$ and behaves as

$$\langle k \rangle = \sum_{k=k_0}^{k_N} k P(k) \approx k_0 \, \frac{(\gamma-1)(1-(k_0/k_N)^{\gamma-2})}{(\gamma-2)(1-(k_0/k_N)^{\gamma-1})} \; . \tag{A.3}$$

This expression has the asymptotic behavior $\langle k \rangle \approx k_0$ for large positive γ corresponding to a random network with uniform vertex degree $k = k_0$. On the other hand, the expression (A.3) implies the mean vertex degree

$$\langle k \rangle \approx k_0 \, \frac{\ln(k_N/k_0)}{1-(k_0/k_N)} \quad \text{for} \quad \gamma = 2 \tag{A.4}$$

which diverges as $\sim \ln k_N$ for large maximal vertex degree k_N.

It is intuitively obvious that the maximal vertex degree must grow with the vertex number N. As explained in the main text, we use the explicit relation

$$k_N = k_0 N^{1/(\gamma-1)} \sim N^{1/(\gamma-1)} \tag{A.5}$$

as in Ref. [1]. In this latter reference, the scaling relation (A.5) was deduced from the requirement that the average number of vertices with vertex degree $k \geq k_N$ is of the order of one, i.e., that

$$N \frac{\sum_{k=k_N}^{\infty} k^{-\gamma}}{\sum_{k=k_0}^{\infty} k^{-\gamma}} \simeq 1 \; . \tag{A.6}$$

Since the scaling relation $k_N \sim N^{1/(\gamma-1)}$ as given by (A.5) affects the N–dependence of our main results, we will now describe an alternative derivation for this scaling relation.

In order to do this, we will start from the scale-free degree distribution $P_\infty(k)$ which is obtained from $P(k)$ as in (A.1) but with $k_N \equiv \infty$. The distribution P_∞ is used to generate N random numbers x_i with $i = 1, 2, ...N$ and $x_i \geq k_0$ which correspond to the vertex degrees k_i of the N vertices i. Since the normalization factor $\mathcal{A}$ as given by (A.2) behaves as $\mathcal{A} \approx k_0^{1-\gamma}/(\gamma-1)$ for large k_N, the random variables x_i are generated according to the probability density

$$P_\infty(x) = (\gamma-1) \, k_0^{\gamma-1} \, x^{-\gamma} \quad . \tag{A.7}$$

The maximal value $x_{\max}$ of the N random numbers x_i is then governed by the probability density

$$\rho(x_{\max}) = N \, (\gamma-1) \, k_0^{\gamma-1} \, x_{\max}^{-\gamma} \left(1 - \left(\frac{k_0}{x_{\max}} \right)^{\gamma-1} \right)^{N-1} \quad . \tag{A.8}$$

It follows from this latter probability density, which is normalized as well, that $x_{\max}$ has the average value

$$\langle x_{\max} \rangle = N k_0 \, B \left(\frac{\gamma-2}{\gamma-1}, N \right) \tag{A.9}$$

where $B(y,N)$ is the standard beta function [2]. In the limit of large network size N, one has $B(y,N) \approx \Gamma(y)\exp(y)N^{-y}$ which implies

$$\langle x_{\max}\rangle \sim k_0 N^{1/(\gamma-1)} \quad , \tag{A.10}$$

i.e., the same k_0– and N–dependence as for k_N in (A.5). The same dependencies are also obtained for the most probable value of $x_{\max}$ which corresponds to the maximum of the distribution $\rho(x_{\max})$ and is given by

$$x_{\max}^{(\mathrm{mp})} = k_0 \left(\frac{1}{\gamma} + \frac{\gamma-1}{\gamma}\, N\right)^{\frac{1}{\gamma-1}} \quad . \tag{A.11}$$

B. Numerical generation of scale-free networks

The random networks of the main text were generated by the so-called configuration model [3] which consists of the following steps:
(i) First, we allocate a vertex degree k_i to each vertex i with $1 \le i \le N$ according to the prescribed degree distribution $P(k)$. We then attach k_i *half-edges* (or 'stubs') to each vertex i. If the sum $\sum_{i=1}^{N} k_i$ is not an even integer, the vertex degree k_i of a randomly chosen vertex i is increased to $k_i + 1$.
(ii) Second, we randomly chose pairs of half-edges and connect them into full edges until the network contains no longer any half-edges (or 'stubs').

Some examples of relatively small networks generated by this configuration model are shown in Fig. 10. In these examples, relatively small values for the minimal vertex degree k_0 were used in order to obtain clear views of the network structure. The networks studied in the main text correspond to larger values of k_0 which leads to rather dense network graphs.

In general, the configuration model may lead to multiple edges between two vertices and to self-connections or self-loops [4]. These configurations have, however, a relativley small statistical weight, and the conclusions of our study remain unchanged if one uses more elaborate schemes that avoid self-connections and multiple edges.

C. Derivation of equation (12)

First, we show that the evolution function $\Psi(Q)$ as defined by Eq. (6) of the main text has a first derivative

$$\Psi'(Q) \equiv \mathrm{d}\Psi(Q)/\mathrm{d}Q \tag{C.1}$$

that behaves as

$$\Psi'(1/2) \sim \sum_k k^{\frac{3}{2}} P(k) \tag{C.2}$$

at the unstable fixed point with $Q = 1/2$.

The explicit form of the evolution function $\Psi(Q)$ is given by

$$\Psi(Q) \equiv \sum_k {\sum_m}' (1 - \frac{1}{2}\delta_{m,k/2})\, k\, P(k) B_{k,m}\, Q^m (1-Q)^{k-m} / \langle k \rangle \tag{C.3}$$

as in equation (6) of the main text with the short hand notation

$$B_{k,m} \equiv \frac{k!}{m!(k-m)!} \equiv \left(\begin{array}{c} k \\ m \end{array} \right) \tag{C.4}$$

for the binomial coefficients. Starting from the explicit form (C.3) for the evolution function, one finds after some computation that

$$\begin{aligned} \Psi'(1/2) &= \sum_{k=1}^{\infty} \sum_{m>k/2}^{k} \frac{kP(k)}{\langle k \rangle} B_{k,m} 2^{1-k} (2m-k) \\ &= \left[\sum_{k=1}^{\infty} \frac{k^2 P(k)}{\langle k \rangle} + \sum_{m=0}^{\infty} \frac{(2m+1)^2 P(2m+1)}{\langle k \rangle 2^{2m}} B_{2m,m} \right] \\ &\quad - \left[\sum_{k=1}^{\infty} \frac{k^2 P(k)}{\langle k \rangle} - \sum_{m=1}^{\infty} \frac{(2m)^2 P(2m)}{\langle k \rangle 2^{2m}} B_{2m,m} \right] \\ &= \sum_{m=0}^{\infty} \frac{(2m+1)^2 P(2m+1)}{\langle k \rangle 2^{2m}} B_{2m,m} + \sum_{m=1}^{\infty} \frac{(2m)^2 P(2m)}{\langle k \rangle 2^{2m}} B_{2m,m} \end{aligned} \tag{C.5}$$

where the identity $(j+1)B_{2j+1,j} = (2j+1)B_{2j,j}$ has been used.

The binomial coefficients can be estimated by Stirling's formula which leads to

$$B_{2m,m} = \frac{(2m)!}{m!m!} \approx \frac{2^{2m}}{\sqrt{\pi}} m^{-1/2} \; . \tag{C.6}$$

If this relation is inserted into (C.5), we obtain

$$\Psi'(1/2) \approx \frac{\sqrt{2/\pi}}{\langle k \rangle} \sum_k k^{\frac{3}{2}} P(k) \tag{C.7}$$

as stated in (C.2).

Finally, we use the explicit form for the degree distribution $P(k)$ as given by (A.1) together with the expressions (A.2) and (A.3) for the normalization factor $\mathcal{A}$ and the mean vertex degree $\langle k \rangle$. When these latter relations are inserted into (C.7), we obtain

$$\Psi'(1/2) \approx \left(\frac{2k_0}{\pi} \right)^{1/2} \frac{(\gamma - 2)}{(\frac{5}{2} - \gamma)} \frac{(k_N/k_0)^{(5-2\gamma)/2} - 1}{1 - (k_0/k_N)^{\gamma - 2}} \tag{C.8}$$

which implies

$$\Psi'(1/2) \approx \left(\frac{k_0}{2\pi} \right)^{1/2} \frac{\ln(k_N/k_0)}{1 - (k_0/k_N)^{1/2}} \quad \text{for} \quad \gamma = 5/2 \tag{C.9}$$

and

$$\Psi'(1/2) \approx 2\left(\frac{2k_0}{\pi}\right)^{1/2} \frac{(k_N/k_0)^{1/2}-1}{\ln(k_N/k_0)} \quad \text{for} \quad \gamma = 2 \quad . \tag{C.10}$$

Finally, the relation $k_N = k_0 N^{1/(\gamma-1)}$ as given by (A.5) implies the asymptotic behavior

$$\begin{aligned} \Psi'(1/2) &\approx 2\sqrt{2k_0/\pi}\, N^{1/2}/\ln(N) && \text{for} \quad \gamma = 2, \\ &\approx \sqrt{2k_0/\pi}\, \tfrac{\gamma-2}{\frac{5}{2}-\gamma}\, N^{\frac{5-2\gamma}{2(\gamma-1)}} && \text{for} \quad 2 < \gamma < 5/2, \\ &\approx \tfrac{1}{3}\sqrt{2k_0/\pi}\ \ln(N) && \text{for} \quad \gamma = 5/2, \text{ and} \\ &\approx \sqrt{2k_0/\pi}\, \tfrac{\gamma-2}{\gamma-\frac{5}{2}} && \text{for} \quad \gamma > 5/2 \end{aligned} \tag{C.11}$$

in the limit of large network size N which is identical to equation (12) of the main text and shows that $\Psi'(1/2)$ diverges in this limit provided $\gamma \leq 5/2$.

D. Derivation of equation (14)

As explained in the main text, the spin patterns on the scale-free networks can be characterized by probabilities q_k that a vertex with degree k is in the spin-up state. The boundary patterns between the two completely ordered spin patterns (or ground states) are characterized by the probabilities $q_k = \hat{q}_k$ and the ordering probability $Q = \hat{Q}$ with

$$\hat{Q} = \sum_k k\, P(k)\, \hat{q}_k / \langle k \rangle = 1/2 \tag{D.1}$$

as in equation (7) of the main text.

Now, let us assume that the system is in the all-spin-down state corresponding to the probabilities $q_k = q_k^- = 0$ for all k and to the ordering probability

$$Q^- = \sum_k k\, P(k)\, q_k^- / \langle k \rangle = 0 \quad . \tag{D.2}$$

We now want to determine the minimal number of spins, $\Omega_{\min}$ that we have to flip in order to reach another pattern which is just 'beyond' a boundary pattern as given by (D.1) and, thus, evolves towards the all-spin-up state under the majority rule dynamics. Comparison of (D.2) and (D.1) suggests that it will be most effective to flip the spins located on the vertices with the highest vertex degrees until one reaches a pattern with $Q \geq 1/2$.

Thus, let us flip the spins on all vertices with a vertex degree k within the range $k_* \leq k \leq k_N$ where the intermediate value k_* represents the lowest vertex degree involved in the spin flips. As a result, we obtain spin patterns that are characterized by probabilities $q_k = q_k^*$ with

$$\begin{aligned} q_k^* &= 0 \qquad \text{for} \quad k_0 \leq k < k_* \\ &= 1 \qquad \text{for} \quad k_* \leq k \leq k_N \end{aligned} \tag{D.3}$$

.

The intermediate value k_* is now determined from the requirement that the corresponding ordering probability $Q = Q^*$ satisfies

$$Q^* = \sum_k k\, P(k)\, q_k^* / \langle k \rangle = \sum_{k=k_*}^{k_N} k\, P(k) / \langle k \rangle = 1/2 \tag{D.4}$$

If we insert the explicit form (A.1) for the degree distribution $P(k)$ into (D.4), we obtain the relation

$$\frac{k_*^{2-\gamma}(1-(k_*/k_N)^{\gamma-2})}{k_0^{2-\gamma}(1-(k_0/k_N)^{\gamma-2})} = 1/2 \tag{D.5}$$

which represents an implicit equation for k_*. The corresponding minimal fraction $\Omega_{\min}$ of flipped spins is given by

$$\Omega_{\min} = \sum_{k \geq k_*}^{k_N} P(k) = \frac{k_*^{1-\gamma}(1-(k_*/k_N)^{\gamma-1})}{k_0^{1-\gamma}(1-(k_0/k_N)^{\gamma-1})} \; . \tag{D.6}$$

Combining the two relations (D.5) and (D.6), we finally obtain the minimal fraction

$$\Omega_{\min} \approx \left(\frac{1 + N^{-(\gamma-2)/(\gamma-1)}}{2} \right)^{(\gamma-1)/(\gamma-2)} \approx 2^{(\gamma-1)/(\gamma-2)} \tag{D.7}$$

in the limit of large N which is identical to equation (14) in the main text. This asymptotic estimate is in very good agreement with and, thus, fully confirmed by the results of numerical simulations, see Fig. 4 of the main text.

E. Hopfield models on scale-free networks

In this appendix, we generalize our mean field analysis to Hopfield models [5, 6] on random scale-free networks. As before, the latter networks are characterized by degree distributions $P(k)$ as given by (A.1). If we denote the graph of such a network by $\mathcal{G}$ and the edge between the vertices i and j by $\langle ij \rangle$, the network's adjacency matrix $\mathbf{I}$ is given by

$$\begin{aligned} I_{ij} &\equiv 1 \qquad \text{for} \quad \langle ij \rangle \in \mathcal{G} \\ &\equiv 0 \qquad \text{otherwise} \end{aligned} \tag{E.1}$$

with $\langle ji \rangle = \langle ij \rangle$ and $I_{ij} = I_{ji}$ for the nondirected networks considered in this appendix. In the original Hopfield model [5], complete graphs have been considered which are characterized by $I_{ij} = 1$ for all possible pairs of vertices i and j. Diluted Hopfield models on random Poissonian networks were previously studied in [6].

Each vertex i of the network has again two internal states described by the Ising spin $\sigma_i = \pm 1$. The total spin pattern at time t will be denoted by $\{\sigma(t)\} \equiv \{\sigma_1(t), \sigma_2(t), \ldots, \sigma_N(t)\}$. In the Hopfield models, the coupling constant J_{ij} between two spins is determined from a set of stored random spin patterns $\{\xi^\mu\} \equiv \{\xi_1^\mu, \xi_2^\mu, \ldots, \xi_N^\mu\}$ with $\mu = 1, 2, \ldots \mathcal{S}$. The variable ξ_i^μ represents the spin value of the random pattern μ on the vertex i and $\xi_i^\mu = +1$ or -1 with equal probability. The total number of stored patterns is denoted by $\mathcal{S}$; the main result of this appendix is an estimate for the maximal possible value of $\mathcal{S}$.

The stored spin patterns $\{\xi^\mu\}$ determine the coupling constant J_{ij} between two spins on vertices i and j via

$$J_{ij} \equiv I_{ij} \frac{1}{\sqrt{\mathcal{S}}} \sum_{\mu=1}^{\mathcal{S}} \xi_i^\mu \xi_j^\mu \quad , \tag{E.2}$$

which leads to the local field

$$h_i(t) = \sum_j J_{ij} \sigma_j(t) = \sum_j I_{ij} \frac{1}{\sqrt{\mathcal{S}}} \sum_{\mu=1}^{\mathcal{S}} \xi_i^\mu \xi_j^\mu \sigma_j(t) \tag{E.3}$$

on spin σ_i at time t. The factor $1/\sqrt{\mathcal{S}}$ is convenient since it leads to coupling constants J_{ij} that stay of the order of one for large $\mathcal{S}$.

This local field determines the time evolution of the spin pattern via the kinetic rule

$$\begin{aligned} \sigma_i(t+1) &= +1 \quad \text{with probability} \quad \mathcal{W}[\beta h_i(t)] \\ &= -1 \quad \text{with probability} \quad \mathcal{W}[-\beta h_i(t)] \end{aligned} \tag{E.4}$$

with the weight function

$$\mathcal{W}(x) \equiv \frac{\exp(x)}{\exp(x) + \exp(-x)} = \frac{1}{1 + \exp(-2x)} \tag{E.5}$$

and the inverse effective temperature $\beta = 1/T_{\text{eff}}$ where this temperature describes the noise intensity in the network.

The overlap $\Lambda(\{\sigma(t)\}, \{\xi^\mu\})$ between the actual spin pattern $\{\sigma(t)\}$ at time t and the stored pattern $\{\xi^\mu\}$ is defined as

$$\Lambda(\{\sigma(t)\}, \{\xi^\mu\}) \equiv \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t)\, \xi_i^\mu \quad . \tag{E.6}$$

We will now assume that, at time $t = 0$, the spin pattern $\{\sigma(0)\}$ on the network has a finite overlap with a certain stored pattern, say $\{\xi^\nu\}$, but has zero overlap with all the other stored patterns. In the following, we will study the time evolution of such a spin pattern.

After we have chosen a certain stored pattern $\{\xi^\nu\}$, the local field h_i as given by (E.3) can be divided up as

$$h_i(t) = \xi_i^\nu \left(\sum_{j=1}^{k} \xi_j^\nu \sigma_j(t) + \sum_{\mu \neq \nu} \sum_{j=1}^{k} \xi_i^\nu \xi_i^\mu \xi_j^\mu \sigma_j(t) \right) / \sqrt{\mathcal{S}} \tag{E.7}$$

for a vertex i with vertex degree k (where $\xi_i^\nu \xi_i^\nu = 1$ has been inserted in the second sum). The first and the second sum in (E.7) contains k and $(\mathcal{S}-1)k$ terms, respectively; each of these terms is equal to ± 1.

If m of the k nearest neighbor spins of σ_i have the values $\sigma_j(t) = -\xi_j^\nu$, the first sum in (E.7) is equal to $k - 2m$. Furthermore, if s of the $(\mathcal{S}-1)k$ terms in the second sum in (E.7) have the value $\xi_i^\nu \xi_i^\mu \xi_j^\mu \sigma_j(t) = -1$, this second sum is equal to $(\mathcal{S}-1)k - 2s$. Thus, for given values of m and s, we obtain the local field

$$h_i(t) = \xi_i^\nu (\mathcal{S}k - 2m - 2s)/\sqrt{\mathcal{S}} \tag{E.8}$$

acting on the spin $\sigma_i(t)$.

In order to determine the probability distribution for the variable m, we now introduce the ordering probability $Q(t)$ that, for any vertex i of the network, a randomly chosen nearest neighbor vertex j is in the spin state $\sigma_j(t) = -\xi_j^\nu$ at time t. This ordering probability represents a generalization of the corresponding probability as defined for the majority rule dynamics in the main text. Note that we use the same symbol Q both for the majority rule dynamics and for the kinetic rule (E.4) which governs the Hopfield models discussed in this appendix.

The variable m varies over the range $0 \leq m \leq k$. The corresponding probability distribution $P_1(m,t)$ at time t can be expressed in terms of the ordering probability $Q(t)$ and is equal to

$$P_1(m,t) = B_{k,m} \left(Q(t)\right)^m \left(1 - Q(t)\right)^{k-m} \tag{E.9}$$

where $B_{k,m}$ denotes the binomial coefficients as in (C.4).

The variable s varies over the range $0 \leq s \leq (\mathcal{S}-1)k$. Since the $\mathcal{S}$ stored random patterns are taken to be statistically independent, it should be a good approximation to assume that the product $\xi_i^\nu \xi_i^\mu \xi_j^\mu \sigma_j(t)$ attains the two possible values ± 1 with equal probability [6, 7]. It then follows that the probability $P_2(s)$ to find a certain value of s is equal to

$$P_2(s) = B_{(\mathcal{S}-1)k,s} / 2^{(\mathcal{S}-1)k} \quad . \tag{E.10}$$

In order to obtain an evolution equation for the ordering probability $Q(t)$, we now introduce the variables $q_k(t)$ which represent the probabilities for a randomly chosen vertex i with degree k to be in the spin state $\sigma_i(t) = -\xi_i^\nu$ at time t. For a random network without vertex–vertex degree correlations, the two quantities $Q(t)$ and $q_k(t)$ fulfill the simple relation

$$Q(t) = \frac{1}{\langle k \rangle} \sum_k k P(k) q_k(t) \quad . \tag{E.11}$$

This relation is identical with equation (3) of the main text even though both the ordering probability Q and the probabilities q_k are now defined with respect to the stored random pattern $\{\xi^\nu\}$.

We now insert the local field as given by (E.8) into the kinetic rule (E.4) and sum over all possible values of m and s which have the joint probability distribution $P_1(m,t)P_2(s)$. In this way, we can calculate the probabilities $q_k(t+1)$ that a k–vertex is in the spin state $\sigma_i(t) = -\xi_i^\nu$ at the next time step. As a result, we obtain

$$q_k(t+1) = \sum_{m=0}^{k} \sum_{s=0}^{(\mathcal{S}-1)k} P_1(m,t)\, P_2(s)\, \mathcal{W}[2\bar{\beta}(m'+s')] \tag{E.12}$$

with the rescaled inverse noise intensity

$$\bar{\beta} \equiv \beta/\sqrt{\mathcal{S}} \tag{E.13}$$

and the shifted indices

$$m' \equiv m - k/2 \quad \text{and} \quad s' \equiv s - (\mathcal{S}-1)k/2 \tag{E.14}$$

and the weight function $\mathcal{W}(x)$ as in (E.5).

Finally, we sum equation (E.12) over all vertex degrees k and use the relation (E.11) between the probabilities q_k and the ordering probability Q which leads to the evolution equation

$$Q(t+1) = \Phi(Q(t)) \tag{E.15}$$

with the evolution function

$$\Phi(Q) \equiv \sum_k \frac{k\, P(k)}{\langle k \rangle\, 2^{(\mathcal{S}-1)k}} \sum_{m=0}^{k} \sum_{s=0}^{(\mathcal{S}-1)k} C(k,m,s) Q^m (1-Q)^{k-m}\, \mathcal{W}[2\bar{\beta}(m'+s')] \tag{E.16}$$

and

$$C(k,m,s) \equiv B_{k,m}\, B_{(\mathcal{S}-1)k,s} = \left(\begin{array}{c} k \\ m \end{array} \right) \left(\begin{array}{c} (\mathcal{S}-1)k \\ s \end{array} \right) \tag{E.17}$$

where we have inserted the expressions (E.9) and (E.10) for the probability distributions $P_1(m,t)$ and $P_2(s)$. The evolution equation (E.15) and the

evolution function (E.16) correspond to the basic equations (5) and (6) in the main text and represent the generalizations of these latter equations to Hopfield models on scale-free networks.

The evolution equation (E.15) has the fixed point $Q = 1/2$ for any degree distribution $P(k)$. This follows from the expression (E.16) for the evolution function $\Phi(Q)$ if one uses the symmetry $B_{k,m} = B_{k,k-m}$ of the binomial coefficients and the identity $\mathcal{W}(x) + \mathcal{W}(-x) = 1$ for the weight function.

The fixed point with $Q = 1/2$ is the *only* fixed point of the evolution equation Eq. (E.15) provided the first derivative $\Phi'(Q) \equiv \mathrm{d}\Phi(Q)/\mathrm{d}Q$ at this fixed point satisfies $\Phi'(1/2) \leq 1$. On the other hand, if

$$\Phi'(1/2) > 1 \quad , \tag{E.18}$$

the fixed point $Q = 1/2$ of Eq. (E.15) becomes unstable and two other stable fixed points appear, one at $Q = Q^- < 1/2$ and the other at $Q = Q^+ > 1/2$. Thus, within mean field theory, the evolution of the binary patterns as governed by the evolution equation (E.15) undergoes a bifurcation at $\Phi'(1/2) = 1$. An analogous bifurcation was previously found for the special case of Hopfield models on random Poissonian networks [6].

The interpretation of this mean field bifurcation is as follows. The fixed point at $Q = 1/2$ corresponds to zero order parameter $y = Q - 1/2 = 0$ and strongly disordered patterns. As long as $\Phi'(1/2) < 1$, the spin pattern $\{\sigma(t)\}$ will evolve towards this disorder fixed point even if it started close to the stored pattern $\{\xi^\nu\}$ initially. Thus, the overlap between the evolving spin pattern $\{\sigma(t)\}$ and the stored pattern $\{\xi^\nu\}$ goes to zero and the memory about the stored pattern is lost. For $\Phi'(1/2) > 1$, on the other hand, the disorder fixed point at $Q = 1/2$ is unstable and the ordering probability $Q(t)$ approaches one of the two other fixed points with nonzero order parameter y. This mean field dynamics represents the time evolution of spin patterns $\{\sigma(t)\}$ which are attracted towards an ensemble of patterns with nonzero overlap with the stored pattern $\{\xi^\nu\}$.

Inspection of the expression (E.16) for $\Phi(Q)$ shows that the quantity $\Phi'(1/2)$ depends on the number $\mathcal{S}$ of the stored spin patterns. In fact, $\Phi'(1/2)$ decreases with increasing $\mathcal{S}$, see below, which implies that the storage criterion $\Phi'(1/2) > 1$ is equivalent to $\mathcal{S} < \mathcal{S}_{\max}$, i.e., to the requirement that the number $\mathcal{S}$ of stored patterns does not exceed a certain maximal number $\mathcal{S}_{\max}$, the storage capacity of the network. In the following, we will estimate this latter quantity.

The first derivative $\Phi'(1/2)$ can be calculated explicitly from the evolution function $\Phi(Q)$ as given by (E.16) . As a result we obtain

$$\Phi'(1/2) = \sum_k \frac{k^2 P(k)}{\langle k \rangle 2^{\mathcal{S}k-1}} \sum_{m=0}^{k-1} \sum_{s=0}^{(\mathcal{S}-1)k} C(k,m,s) \tanh[\bar{\beta}(1 + 2m'' + 2s')] \tag{E.19}$$

with the shifted variable

$$m'' \equiv m - (k-1)/2 \tag{E.20}$$

and s' as in (E.14).

We now want to explore the possibility to store a large number $\mathcal{S}$ of random patterns and, thus, study the behavior of $\Phi'(1/2)$ as given by (E.19) in the limit of large $\mathcal{S}$. Replacing the binomial distributions by normal distributions and using $\bar{\beta} = \beta/\sqrt{\mathcal{S}}$ as in (E.13), we obtain

$$\Phi'(1/2) \approx \sum_k \frac{k^2 P(k)}{\langle k \rangle} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{\pi}} \exp(-z^2) \tanh[\beta\sqrt{2k}z + \beta/\sqrt{\mathcal{S}})] \equiv f(\beta, \mathcal{S}) \quad . \tag{E.21}$$

The integral in this expression depends on two variables, the inverse noise intensity β and the storage capacity $\mathcal{S}$. In the low noise limit corresponding to large β, the tanh–function becomes piece-wise constant and one obtains

$$\Phi'(1/2) \approx f(\infty, \mathcal{S}) = \sum_k \frac{k^2 P(k)}{\langle k \rangle} \operatorname{erf}(1/\sqrt{2k\mathcal{S}}) \tag{E.22}$$

with the error function

$$\operatorname{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y \mathrm{d}z \, \exp(-z^2) \quad . \tag{E.23}$$

For large $\mathcal{S}$ or small $y \equiv 1/\sqrt{2k\mathcal{S}}$, this leads to

$$\Phi'(1/2) \approx \left(\frac{2}{\pi\mathcal{S}}\right)^{1/2} \sum_k \frac{k^{3/2} P(k)}{\langle k \rangle} = \frac{1}{\sqrt{\mathcal{S}}} \Psi'(1/2) \tag{E.24}$$

with $\Psi'(1/2)$ as in (C.7).

Finally, we insert the asymptotic form (E.24) into the storage criterion $\Phi'(1/2) > 1$ which leads to the inequality $\mathcal{S} < \mathcal{S}_{\max}$ with the storage capacity

$$\mathcal{S}_{\max} \approx [\Psi'(1/2)]^2 \quad . \tag{E.25}$$

It then follows from the asymptotic behavior of $\Psi'(1/2)$ as given by (C.8) that the storage capacity behaves as

$$\begin{aligned} \mathcal{S}_{\max} &\sim k_0 \, N/\ln^2(N) && \text{for} \quad \gamma = 2, \\ &\sim k_0 \, N^{(5-2\gamma)/(\gamma-1)} && \text{for} \quad 2 < \gamma < 5/2, \text{ and} \\ &\sim k_0 \, \ln^2(N) && \text{for} \quad \gamma = 5/2 \end{aligned} \tag{E.26}$$

in the limit of large network size N and low noise intensity corresponding to large β.

The above analysis for $\gamma \leq 5/2$ is selfconsistent since, in this case, $\mathcal{S}_{\max}$ becomes large for large N, and it is then justified to replace the relation (E.22) by its asymptotic form (E.24) which is valid for large $\mathcal{S}$. This analysis can be extended to arbitrary values of the inverse noise intensity β. One then finds that $\mathcal{S}_{\max}$ decreases monotonically with decreasing β but that the asymptotic behavior as given by (E.26) is still valid for $\beta > 0$ in the limit of large N.

For $\gamma > 5/2$, on the other hand, the asymptotic form (E.24) of $\Phi'(1/2)$ leads to the maximal storage capacity $\mathcal{S}_{\max} \sim k_0$ which does *not* grow with N for large N. The same conclusion is drawn if one uses the full expression (E.22) in order to determine $\mathcal{S}_{\max}$ in the low noise limit corresponding to large β. Furthermore, $\mathcal{S}_{\max}$ decreases again monotonically with decreasing β. Thus, for $\gamma > 5/2$, Hopfield models on scale-free networks can only store a *finite* number of patterns even in the limit of large network size N. This latter conclusion is consistent with previous results on random Poissonian networks [6].

So far, we have discussed the capability of the network to retrieve single stored patterns. We have also studied the capability of the network to discriminate between two or more stored patterns. At time $t = 0$, the spin pattern is taken to have nonzero overlap with two stored random patterns $\{\xi^{\nu_1}\}$ and $\{\xi^{\nu_2}\}$ and zero overlap with all other stored patterns. The spin pattern will then evolve towards the fixed point or limit cycle which corresponds to the stored pattern with the larger initial overlap.

F. Extension to directed scale-free networks

In many real-world networks, the interactions between the vertices are asymmetric and the links or edges are *directed* which implies that these networks are characterized, in general, by different in-degree and out-degree distributions. In a gene regulation network, for example, the product of gene A may regulate the expression of gene B, but the product of gene B may not regulate the expression of gene A. It is not difficult to extend the majority rule dynamics as defined by equation (2) of the main text to directed networks by using an asymmetric adjacency matrix. As explained in this Appendix, we find that $\gamma = 5/2$ again provides a sharp boundary for the pattern evolution provided (i) the in-degree and the out-degree distributions are both scale-free and (ii) the in-degree and the out-degree of the vertices are correlated.

In order to extend the majority rule dynamics to directed networks, we introduce the asymmetric adjacency matrix $\mathbf{D}$ with matrix elements $D_{ij} = 1$

if the vertices i and j are nearest neighbors and the directed edge between them points towards vertex i. The majority dynamics is then governed by the local fields $h_i \equiv \sum_j D_{ij}\sigma_j$.

Now, consider the probability $Q_{\rm up}(t)$ that a randomly chosen up–stream nearest-neighbor is in the spin-up state at time t. For a random network without correlations between the degrees of different vertices, we obtain the relation

$$Q_{\rm up}(t) = \sum_{k_{\rm out}} \frac{k_{\rm out} P_{\rm out}(k_{\rm out})}{\langle k \rangle_{\rm out}} q(k_{\rm out}, t) \ , \tag{F.1}$$

where $P_{\rm out}(k_{\rm out})$ is the vertex out-degree distribution, $\langle k \rangle_{\rm out}$ is the mean vertex out–degree, and $q(k_{\rm out}, t)$ is the probability that a vertex of out–degree $k_{\rm out}$ has its spin pointing up at time t.

The probability $Q_{\rm up}(t)$ is again governed by an evolution equation of the form $Q_{\rm up}(t+1) = \Psi_{\rm up}(Q_{\rm up}(t))$. The explicit form of the evolution function $\Psi_{\rm up}(Q_{\rm up})$ can be obtained by the same analytical procedure used to derive equation (C.3), which is identical with Equation (6) of the main text, for nondirected networks. If both the minimal in–degree and the minimal out–degree are taken to be larger than zero, one has $\Psi_{\rm up}(0) = 0$, $\Psi_{\rm up}(1) = 1$, and $\Psi_{\rm up}(1/2) = 1/2$, and the evolution equation for $Q_{\rm up}(t)$ has again three fixed points, two stable ones at $Q_{\rm up} = 0$ and $Q_{\rm up} = 1$, and one unstable one at $Q_{\rm up} = 1/2$. The slope of the evolution function $\Psi_{\rm up}(Q_{\rm up})$ at the unstable fixed point scales as

$$\Psi'_{\rm up}(1/2) \sim \sum_{k_{\rm out}} k_{\rm out}\, P_{\rm out}(k_{\rm out}) \sum_{k_{\rm in}} k_{\rm in}^{1/2}\, P(k_{\rm in}|k_{\rm out}) \ , \tag{F.2}$$

where $P(k_{\rm in}|k_{\rm out})$ is the conditional probability that the in-degree of a randomly chosen vertex is $k_{\rm in}$ provided its out-degree is $k_{\rm out}$.

As a first example, consider a network for which (i) the out-degree distribution is Poissonian, i.e.,

$$P_{\rm out}(k_{\rm out}) = (\langle k \rangle_{\rm out})^{k_{\rm out}} e^{-\langle k \rangle_{\rm out}} / k_{\rm out}! \quad , \tag{F.3}$$

(ii) the in-degree distribution is scale-free, i.e.,

$$P_{\rm in}(k_{\rm in}) \sim k_{\rm in}^{-\gamma} \quad , \tag{F.4}$$

and (iii) there is no correlation between the in-degree and out-degree of a vertex, i.e.,

$$P(k_{\rm in}|k_{\rm out}) \equiv P_{\rm in}(k_{\rm in}) \quad . \tag{F.5}$$

For such a directed network, the relation (F.2) implies that $\Psi'_{\rm up}(1/2)$ is of order unity for $\gamma > 3/2$ in the limit of large network size N but diverges with N if $\gamma < 3/2$. This distinction is consistent with the recent study in [8].

The form as given by Eq. (F.2) implies that the response behavior of the directed network is fast provided (i) the in-degree and out-degree distributions are both scale-free, and (ii) the in- and out-degrees of the same vertices are positively correlated.

As a second example, consider networks for which both the in-degree and the out-degree distribution are scale-free and the conditional probability behaves as

$$P(k_{\rm in}|k_{\rm out}) \sim (k_{\rm in} + k_{\rm out})^{-\gamma-1} \quad . \tag{F.6}$$

For $\gamma \leq 5/2$, this leads again to a divergence of $\Psi'_{\rm up}(1/2)$ in the limit of large network size N. On the other hand, if the in- and out-degrees are uncorrelated, the quantity $\Psi'_{\rm up}(1/2)$ will remain finite in the large N limit as long as $\gamma > 2$. Therefore, a directed network can respond rapidly if it exhibits positive correlations between the in-degree and the out-degree in addition to its scale-free degree distributions.

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Supporting Figures

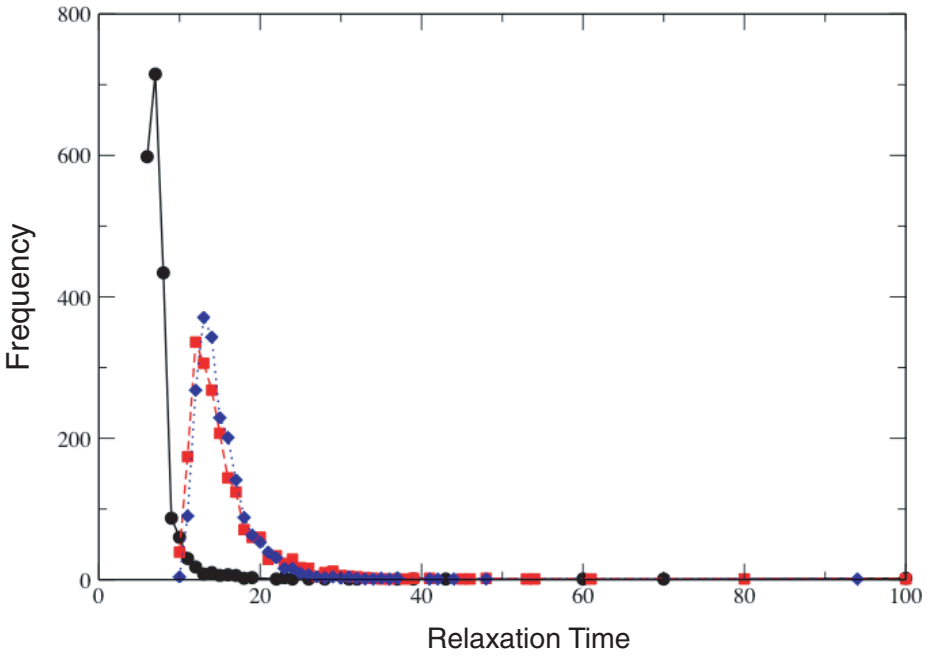

Figure 5: Distribution of relaxation times for the same random networks as in Fig. 1 and Fig. 2: (i) scale-free with $\gamma = 2.25$ and $k_0 = 5$ (circles), (ii) scale-free with $\gamma = 3$ and $k_0 = 10$ (squares), and (iii) Poissonian (diamonds). All three networks have the same mean vertex degree $\langle k \rangle = 20$. For each network, 2000 individual trajectories have been generated, each starting from a strongly disordered pattern as defined by equation (9) of the main text. For each of these trajectories, the relaxation time is equal to the number of time steps until the trajectory has reached one of the two completely ordered patterns. The probability to observe a relaxation time that exceeds 100 steps is smaller than 10^{-2} in all three cases.

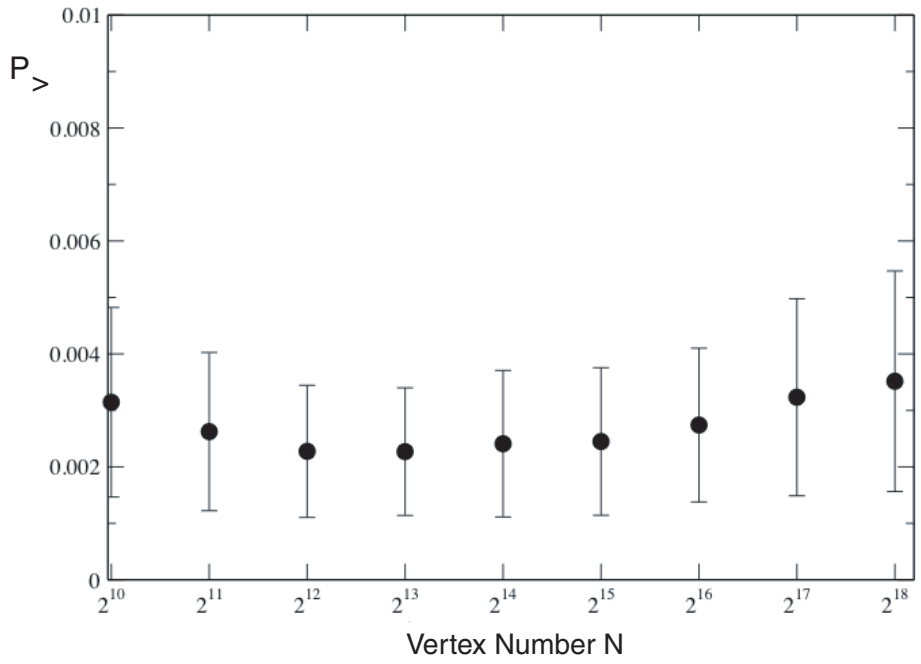

Figure 6: Probability $P_>$ of not reaching one of the two completely ordered patterns within 100 time steps as a function of vertex number (or network size) N. Each data point corresponds to an average of 100 different networks with $\gamma = 2.25$ and $k_0 = 5$. For each network, 2000 individual pattern trajectories have been generated starting from a strongly disordered patterns as defined by Eq. (9) of the main text.

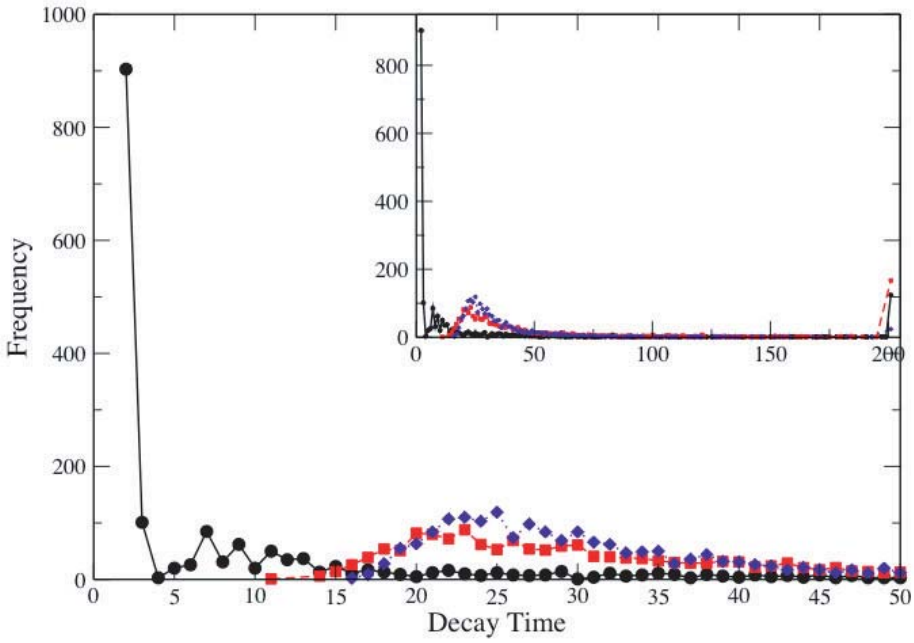

Figure 7: Distribution of decay times for three different random networks with $N = 2^{18}$ vertices and mean vertex degree $\langle k \rangle = 10$. For each network, 2000 individual pattern trajectories have been generated starting from a strongly disordered pattern as defined by equation (9) of the main text. The decay time is equal to the number of time steps until a pattern with $|Q - 1/2| \geq 1/4$ is reached. The three networks are (i) scale-free with $\gamma = 2.148$ and $k_0 = 2$ (circles); (ii) scale-free with $\gamma = 2.828$ and $k_0 = 5$ (squares); and (iii) Poissonian (diamonds). The inset shows the same data but with the horizontal axis extended to 200 decay steps.

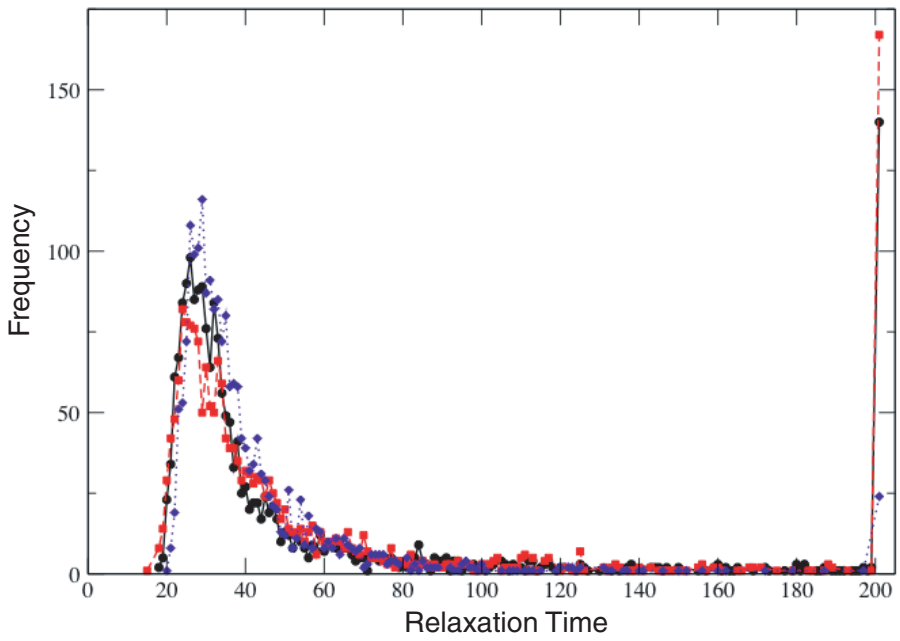

Figure 8: Distribution of relaxation times for the same networks and for the same ensembles of trajectories as in Figure 7. The relaxation time is equal to the number of time steps until one of the two ordered patterns has been reached. In contrast to the distribution of the decay times as shown in Figure 7, the distribution of the relaxation times is similar for all three networks. This implies that, for this realization of the scale-free network with $\gamma = 2.148$ (circles), the approach towards the two completely ordered states is slowed down for larger times, see Figure 8.

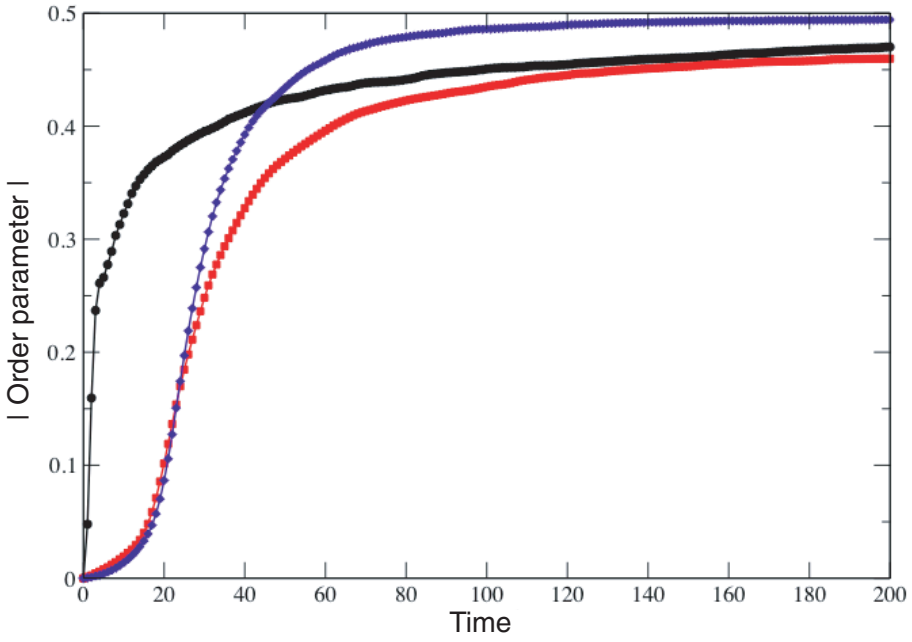

Figure 9: Absolute value of the order parameter, $|y| = |Q - \frac{1}{2}|$ as a function of time (in units of iteration steps) for the same three networks as in Figures 7 and 8. Each data set represents an average over 2000 individual trajectories which all start initially from strongly disordered spin patterns. For the scale-free network with $\gamma = 2.148$ (circles), the absolute value of the order parameter, $|y|$, quickly exceeds the value $|y| = 1/4$ but the subsequent approach towards $|y| = 1$ is slower than in the case of the Poissonian network (diamonds).

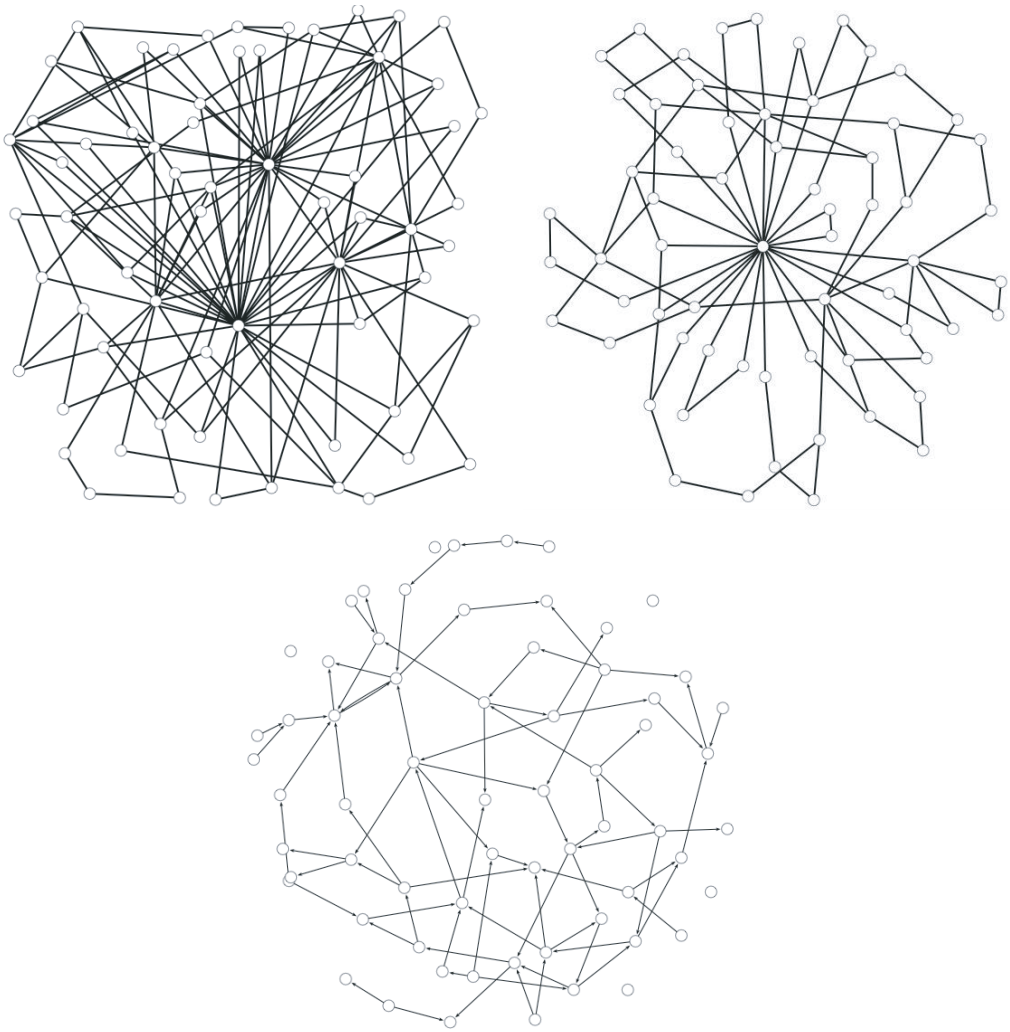

Figure 10: Three networks obtained via the configuration model: (Upper left) random scale-free network with $N = 64$ vertices and $M = 140$ edges generated from a scale-free vertex distribution with $\gamma = 2.5$ and $k_0 = 2$; (Upper right) random scale-free network with $N = 64$ vertices and $M = 96$ edges generated from a scale-free vertex degree distribution with $\gamma = 3$ and $k_0 = 2$; (Bottom) random Poissonian network with $N = 64$ vertices and $M = 82$ edges.