Activity Patterns on Scale-Free Networks

Reinhard Lipowsky 11.11.1953
1978: Diploma, Physics,
Thesis with Heinz Horner on
turbulence (University of Heidelberg)
1982: PhD (Dr. rer. nat.), Physics
(University of Munich)
Thesis with Herbert Wagner
on surface phase transitions
1979-1984: Teaching Associate with
Herbert Wagner (University of Munich)
1984-1986: Research Associate with
Michael E. Fisher (Cornell University)
1986-1988: Research Associate with
Heiner Müller-Krumbhaar (FZ Jülich)
1987: Habilitation, Theoretical Physics
(University of Munich)
Thesis: Critical behavior of interfaces:
Wetting, surface melting and related
phenomena
1989-1990: Associate Professorship
(University of Munich)
1990-1993: Full Professorship
(University of Cologne), Director of
the Division "Theory II" (FZ Jülich)
Since Nov 1993: Director
(Max Planck Institute of Colloids
and Interfaces, Potsdam)

The biosphere contains many complex networks built up from rather different elements such as molecules, cells, organisms, or machines. In spite of their diversity, these networks exhibit some univeral features and generic properties. The basic elements of each network can be represented by nodes or vertices. Furthermore, any binary relation between these elements can be described by connections or edges between these vertices as shown in **Fig. 1**. By definition, the degree k of a given vertex is equal to the number of edges connected to it, i.e., to the number of direct neighbors. Large networks containing many vertices can then be characterized by their degree distribution, P(k), which represents the probability that a randomly chosen vertex has degree k.

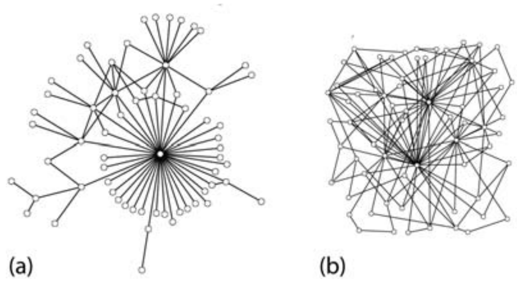

Fig. 1: Two examples for small scale-free networks: (a) Network with scaling exponent $\gamma = 2$ and minimal degree $k_0 = 1$. This network has a tree-like structure and a small number of closed cycles; and (b) Network with scaling exponent $\gamma = 5/2$ and minimal degree $k_0 = 2$ for which all edges belong to closed cycles.

Scale-Free Degree Distributions

Many biological, social, and technological networks are found to be scale-free in the sense that their degree distribution decays as

$$P(k) \sim 1/k^{\gamma} \text{ for } k > k_0$$

which defines the scaling exponent γ. Typical values for this exponent are found to lie between 2 and 5/2. **[1,2]** As one would expect naively, there are fewer vertices with a larger number of connections. However, since the probability P(k) decreases rather slowly with k, a large network with many vertices always contains some high-degree vertices with a large number of direct neighbors.

As an example, let us consider neural networks. The human brain consists of about 100 billion nerve cells or neurons that are interconnected to form a huge network. Each neuron can be active by producing an action potential. If we were able to make a snapshot of the whole neural network, we would see, at any moment in time, a certain pattern of active and inactive neurons. If we combined many such snapshots into a movie, we would find that this activity pattern changes continuously with time. At present, one cannot observe such activity patterns on the level of single neurons, but modern imaging techniques enable us to monitor coarse-grained patterns with a reduced spatial resolution. Using functional magnetic resonance imaging, for example, we can obtain activity patterns of about 100 000 neural domains, each of which contains about a million neurons.

These neural domains form another, coarse-grained network. Each domain corresponds to a vertex of this network, and each vertex can again be characterized by its degree k, i.e., by the number of connections to other vertices. It has been recently concluded from magnetic resonance images that the functional networks of neural domains are scale-free and characterized by a degree distribution with scaling exponent $\gamma = 2.1$.

Dynamical Variables and Activity Patterns

In general, the elements of real networks are dynamic and exhibit various properties that change with time. A more detailed description of the network is then obtained in terms of dynamical variables associated with each vertex of the network. In many cases, these variables evolve fast compared to changes in the network topology, which is therefore taken to be time-independent. Two examples for such dynamical processes are provided by neural networks that can be characterized by firing and nonfiring neurons or by the regulation of genetic networks that exhibit a changing pattern of active and inactive genes. In these examples, each dynamical variable can attain only two states (active or inactive), and the configuration of all of these variables defines the activity pattern of the network as shown in **Fig. 2**.

Fig. 2: Three subsequent snapshots of the activity pattern on a small scale-free network with 31 vertices and 50 edges. The active and inactive vertices are yellow and blue, respectively. For the initial pattern on the left, about half of the vertices are inactive (blue); for the final pattern on the right, almost all vertices are active (yellow). Each vertex of the network has a certain degree which is equal to the number of connections attached to it; this number is explicitly given for some nodes on the left.

Local Majority Rules Dynamics

In collaboration with Haijun Zhou (now professor at ITP, CAS, Beijing), we have recently started to theoretically study the time evolution of such activity patterns. **[3,4]** We focused on the presumably simplest dynamics as generated by a local majority rule: If, at a certain time, most direct neighbors of a certain vertex are active or inactive, this vertex will become active or inactive at the next update of the pattern. This dynamical rule leads to two fixed points corresponding to two completely ordered patterns, the all-active pattern and the all-inactive one. Each fixed point has a basin of attraction consisting of all patterns that evolve towards this fixed point for sufficiently long times. The boundary between the two basins of attraction of the two fixed points represents the so-called separatrix. One global characterization of the space of activity patterns is the distance of a fixed point from the separatrix as measured by the smallest number of vertices one has to switch from active to inactive (or vice versa) in order to reach the basin of attraction of the other fixed point.

Distance Between Fixed Points and Separatrix

We found that, for scale-free networks, this distance corresponds to selective switches of the high-degree vertices and strongly depends on the scaling exponent γ. For a network with N vertices, the number Ω of highly connected vertices that one has to switch in the all-active (or all-inactive) pattern in order to perturb this pattern beyond the separatrix grows as $\Omega = N/2^{\zeta}$ with $\zeta = (\gamma-1)/(\gamma-2)$ and vanishes as an essential singularity when the scaling exponent γ approaches the value $\gamma = 2$ from above. **[3]** If we used random rather than selective switches, on the other hand, we would have to switch of the order of N/2 vertices irrespective of the value of γ. Note that, in the limit in which the scaling exponent γ becomes large, selective and random switching lead to the same distance Ω. A low-dimensional cartoon of the high-dimensional pattern space is shown in **Fig. 3**.

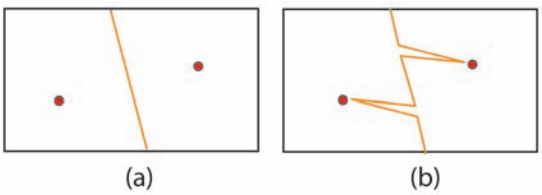

Fig. 3: Two fixed points (red dots) and separatrix (orange line) between their basins of attraction; (a) For large values of the scaling exponent γ the separatrix is smooth; (b) As the scaling exponent is decreased towards the value $\gamma = 2$, the separatrix develops spikes which come very close to the fixed points. These spikes correspond to the selective switching of the high-degree vertices.

Decay Times of Disordered Patterns

Another surprising feature of activity patterns on scale-free networks is the evolution of strongly disordered patterns that are initially close to the separatrix. These patterns decay towards one of the two ordered patterns but the corresponding decay time, i.e., the time it takes to reach these fixed points, again depends strongly on the scaling exponent γ.

We have developed a mean field theory that predicts qualitatively different behavior for $\gamma < 5/2$ and $\gamma > 5/2$. **[3,4]** For $2 < \gamma < 5/2$, strongly disordered patterns decay within a finite decay time even in the limit of infinite networks. For $\gamma > 5/2$, on the other hand, this decay time diverges logarithmically with the network size N. These mean field predictions have been checked by extensive computer simulations of two different ensembles of random scale-free networks using both parallel (or synchronous) as well as random sequential (or asynchronous) updating. **[4]** The two ensembles consist of (i) multi-networks that typically contain many self-connections and multiple edges and (ii) simple-networks without self-connections and multiple edges. For simple-networks, the simulations confirm the mean field results, see **Fig. 4**. For multi-networks, it is more difficult to determine the asymptotic behavior for large number of vertices since these networks are governed by an effective, N-dependent scaling exponent γ_{eff} that exceeds γ for finite values of N. **[4]**

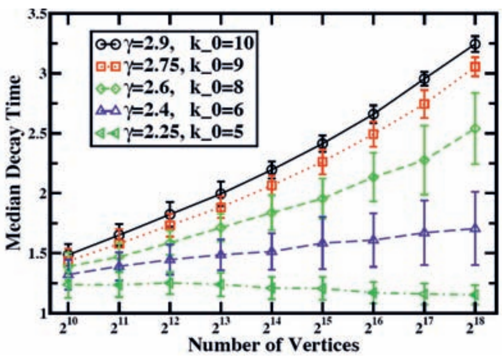

Fig. 4: Decay times for strongly disorderd patterns as a function of the number, N, of vertices contained in simple-networks for random sequential updating. The minimal vertex degree k_0 was chosen in such a way that the average degree is roughly equal for all values of the scaling exponent γ In the limit of large N, the decay times attain a finite value for $\gamma < 5/2$ but increase logarithmically with N for $\gamma > 5/2$.

R. Lipowsky, J. Menche, A. Valleriani
reinhard.lipowsky@mpikg.mpg

References:

[1] Albert, R. and Barabasi, A.-L.: Statistical mechanics of complex networks. Rev. Mod. Phys. **74**, 47 (2002).

[2] Newman, M. E. J.: The structure and function of complex networks. SIAM Review **45**, 167 (2003).

[3] Zhou, H., and Lipowsky, R.: Dynamic pattern evolution on scale-free networks. PNAS **102**, 10052 (2005).

[4] Zhou, H., and Lipowsky, R.: Activity patterns on random scale-free networks: Global dynamics arising from local majority rules. J. Stat. Mech: Theory and Experiment **2007**, 01009 (2007).